Stochastic Structural Dynamics

Lecture-8

Random processes-3

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Recall

$$\boxed{\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(i\omega\tau)\,d\tau \\ R_{XX}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega \end{aligned}}$$

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Gaussian random process

Let $X(t)$ be a random process and consider its 1st and 2nd order pdf-s.

$$p_X(x;t) = \frac{1}{\sqrt{2\pi}\sigma_X(t)} \exp\left[-\frac{1}{2}\left\{\frac{x - m_X(t)}{\sigma_X(t)}\right\}^2\right]; -\infty < x < \infty$$

$$p_{XX}(x_1, x_2; t_1, t_2) = \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{\left[1 - r_{12}^2\right]}}$$

$$\exp\left[-\frac{1}{2\left\{1 - r_{12}^2\right\}}\left\{\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2} - 2r_{12}\frac{(x_1 - m_1)(x_2 - m_2)}{\sigma_1\sigma_2}\right\}\right]$$

$$-\infty < x_1, x_2 < \infty$$

$$m_1 = m_X(t_1); m_2 = m_X(t_2); \sigma_1 = \sigma_X(t_1); \sigma_2 = \sigma_X(t_2); r_{12} = r_{XX}(t_1, t_2)$$

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Continuing further, consider n time instants $\left\{t_i\right\}_{i=1}^{n}$ and associated random variables $\left\{X\left(t_i\right)\right\}_{i=1}^{n}$.

Let the jpdf of $\left\{X\left(t_i\right)\right\}_{i=1}^{n}$ be given by

$$p_{XX\cdots X}\left(x_1,x_2,\cdots,x_n;t_1,t_2,\cdots,t_n\right)=$$

$$\frac{1}{\left(2\pi\right)^{\frac{n}{2}}\left|S\right|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(x-\eta\right)^t S^{-1}\left(x-\eta\right)\right];-\infty<x_i<\infty\forall i\in\left[1,n\right]$$

$$S_{ij}=\left\langle\left[X\left(t_i\right)-m_X\left(t_i\right)\right]\left[X\left(t_j\right)-m_X\left(t_j\right)\right]\right\rangle$$

$Note:$ $S^t=S$ & S is positive definite.

$$\eta=\left[m_X\left(t_1\right)\quad m_X\left(t_2\right)\quad\dots\quad m_X\left(t_n\right)\right]^t$$

$$x=\left[x_1\quad x_2\quad\cdots\quad x_n\right]$$

Definition

$X\left(t\right)$ is said to be a Gaussian random process if the above form of pdf is true for any n and for any choice of $\left\{t_i\right\}_{i=1}^{n}$.

Remarks

(a) A Gaussian random process is completely specified through its mean $m_X(t)$ and covariance $C_{XX}(t_1,t_2)$.

(b) $X(t)$ is stationary $\Rightarrow m_X(t) = m_X$ & $C_{XX}(t_1,t_2) = C_{XX}(t_1 - t_2)$

$\Rightarrow p_{XX}(x_1,x_2;t_1,t_2) = p_{XX}(x_1,x_2;t_1 - t_2)$

$\Rightarrow X(t)$ is 2nd order SSS $\Rightarrow X(t)$ is SSS.

(c) A stationary Gaussian random process with zero mean is completely described by its autocovariance function or its pdf function.

(d) Linear transformation of Gaussian random processes preserve the Gaussian nature. Gaussian distributed loads on linear systems produce Gaussian distributed responses.

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Fourier representation of a Gaussian random process

Let $X(t)$ be a zero mean, stationary, Gaussian random process defined as

$$X(t) = \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t; \;\; \omega_n = n\omega_0$$

Assumptions

Here $a_n \sim N(0, \sigma_n), b_n \sim N(0, \sigma_n),$

$\langle a_n a_k \rangle = 0 \forall n \neq k, \langle b_n b_k \rangle = 0 \forall n \neq k,$

$\langle a_n b_k \rangle = 0 \forall n, k = 1, 2, \cdots, \infty$

$$\Rightarrow \langle X(t) \rangle = \sum_{n=1}^{\infty} \left\{ \langle a_n \rangle \cos \omega_n t + \langle b_n \rangle \sin \omega_n t \right\} = 0$$

$$\left\langle X(t)X(t+\tau)\right\rangle = \left\langle \sum_{n=1}^{\infty}\left\{a_n\cos\omega_n t + b_n\sin\omega_n t\right\}\sum_{n=1}^{\infty}\left\{a_n\cos\omega_n(t+\tau) + b_n\sin\omega_n(t+\tau)\right\}\right\rangle$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\langle\left(a_n\cos\omega_n t + b_n\sin\omega_n t\right)\left(a_m\cos\omega_m(t+\tau) + b_m\sin\omega_m(t+\tau)\right)\right\rangle$$

$$= \sum_{n=1}^{\infty}\langle a_n^2\rangle\cos\omega_n t\cos\omega_n(t+\tau) + \langle b_n^2\rangle\sin\omega_n t\sin\omega_n(t+\tau)$$

$$\Rightarrow R_{XX}(\tau) = \sum_{n=1}^{\infty}\sigma_n^2\cos\omega_n\tau$$

$\langle a_n^2\rangle = \sigma_n^2 \quad \langle b_n^2\rangle = \sigma_n^2$

$X(t)$ is a WSS random process.

$X(t)$ is Gaussian.

$\Rightarrow X(t)$ is a SSS process.

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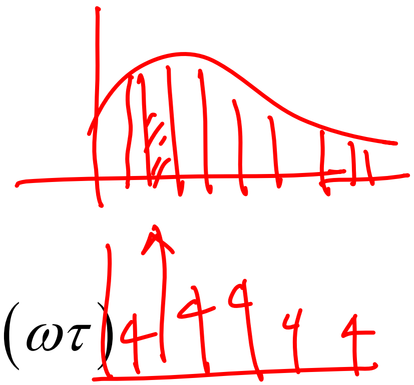
Fourier representation of a Gaussian random process (continued)

Consider the psd function



$$S_{XX}(\omega)=\sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)$$

$$\Rightarrow \tilde{R}_{XX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)\cos(\omega\tau)$$

$$\Rightarrow \tilde{R}_{XX}(\tau)=\frac{1}{2\pi}\sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n\cos(\omega_n\tau)$$

Compare this with

$$R_{XX}(\tau)=\sum_{n=1}^{\infty}\sigma_n^2\cos\omega_n\tau$$

By choosing $\sigma_n^2=\dfrac{S(\omega_n)\Delta\omega_n}{2\pi}$, we see that the two ACF-s coincide.

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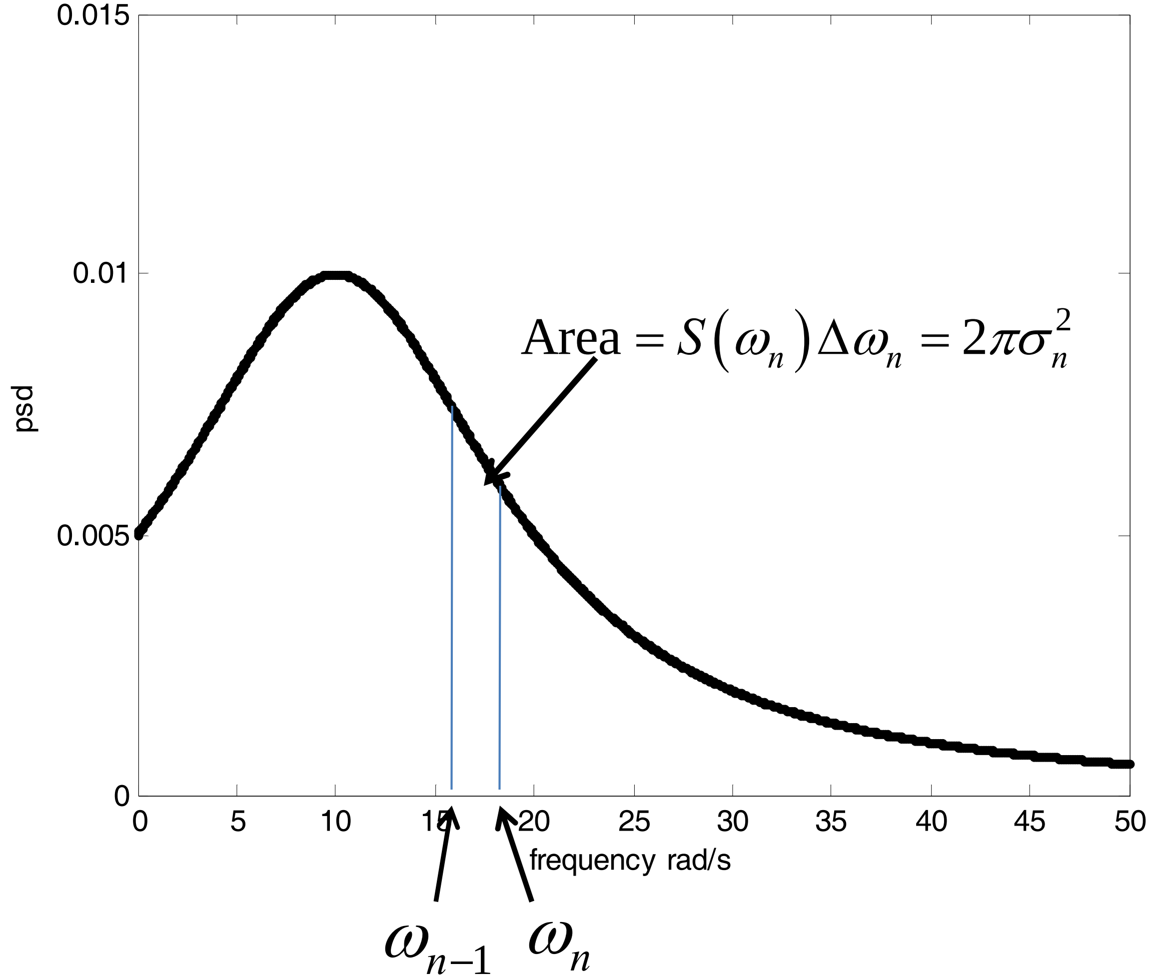
By discretizing the psd function as shown we can simulate samples of $X(t)$ using the Fourier representation

$$X\left(t\right)=\sum_{n=1}^{\infty}\left\{a_n \cos\omega_n t+b_n \sin\omega_n t\right\};\ \ \omega_n=n\omega_0$$



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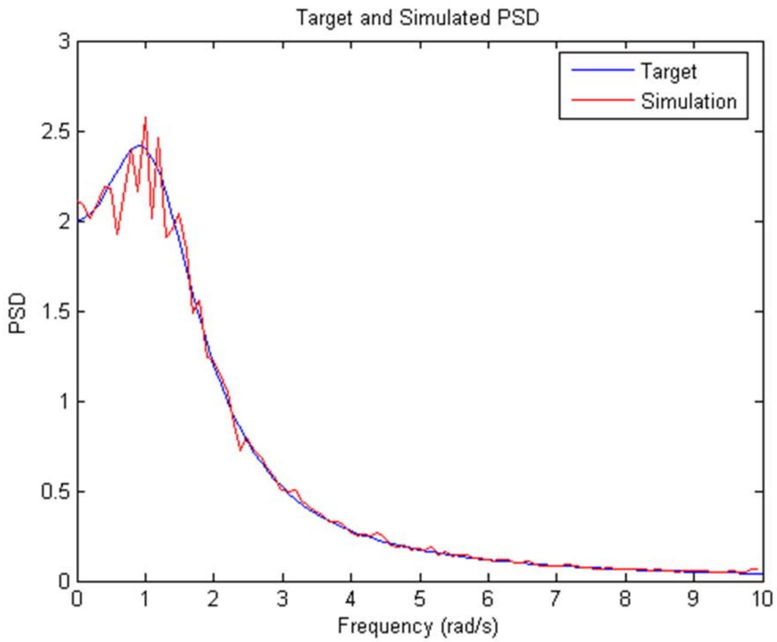
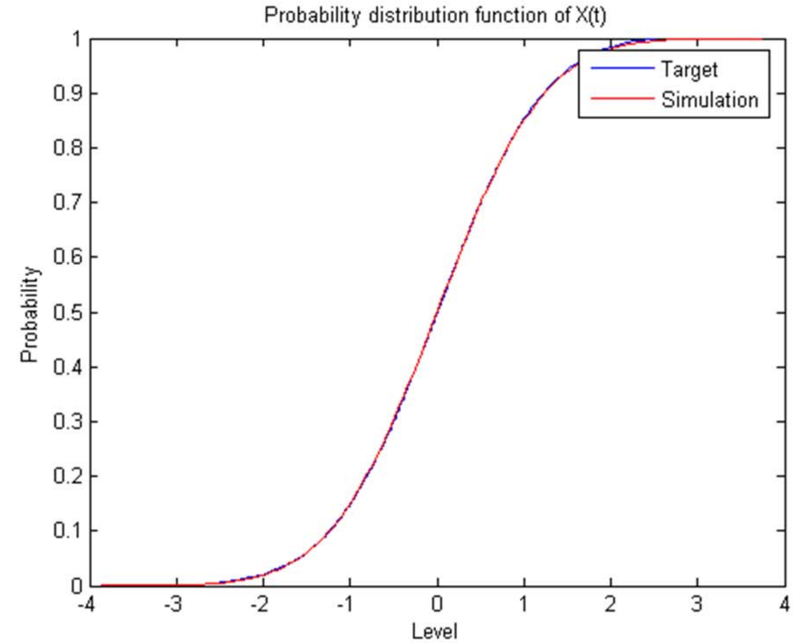
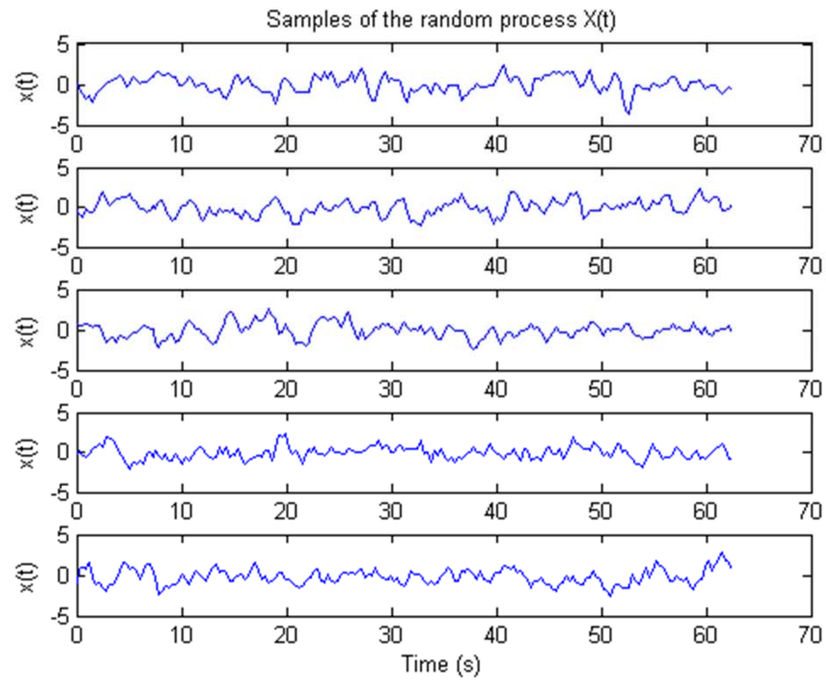




Ensemble of realizations of random processes can be digitally simulated



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Simple random walk

Let $\{X_i\}_{i=1}^{\infty}$ be an iid sequence of random variables with

$$\mathrm{P}(X = \Delta x) = p$$

$$\mathrm{P}(X = -\Delta x) = q$$

such that $p + q = 1.$

$$\langle X \rangle = \mathrm{P}(X = \Delta x)(\Delta x) + \mathrm{P}(X = -\Delta x)(-\Delta x)$$

$$= \Delta x (p - q)$$

$$\langle X^2 \rangle = \mathrm{P}(X = \Delta x)(\Delta x)^2 + \mathrm{P}(X = -\Delta x)(-\Delta x)^2$$

$$= \Delta x^2 (p + q)$$

$$\mathrm{Var}(X) = \langle X^2 \rangle - \langle X^{\cancel{2}} \rangle^2$$

$$= \Delta x^2 (p + q) - \Delta x^2 (p - q)^2$$

$$= \Delta x^2 (p + q)^2 - \Delta x^2 (p - q)^2 \quad (\because p + q = 1)$$

$$= \Delta x^2 \left[(p + q)^2 - (p - q)^2 \right] = 4pq\Delta x^2$$

Let t be the time axis and let us divide the interval $(0,t)$ into n subintervals each of width Δt such that

$$n\Delta t = t.$$

Define

$$S(t) = \sum_{i=1}^{n} X_i$$

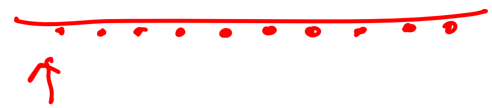
$$\Rightarrow \langle S(t) \rangle = \sum_{i=1}^{n} \langle X_i \rangle = \sum_{i=1}^{n} (p-q)\Delta x$$

$$= n(p-q)\Delta x$$

$$= t(p-q)\frac{\Delta x}{\Delta t}$$

$$\mathrm{Var}\left[S(t)\right] = t4pq\Delta x^2$$

$$= t4pq\frac{\Delta x^2}{\Delta t}$$



Remarks

- $S(t)$ is known as a simple random walk.
- $S(t)$ is a discrete state, discrete parameter random process.
- Consider the limit of $\Delta x \to 0$ as $\Delta t \to 0$

$\Rightarrow$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \langle S \rangle = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} t(p-q)\frac{\Delta x}{\Delta t}$$

and

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \mathrm{Var}\left[S(t)\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} t4pq\frac{\Delta x^2}{\Delta t} \to 0$$

$\Rightarrow$

In the limit of $\Delta x \to 0$ as $\Delta t \to 0$, $S(t)$ becomes a deterministic function.

This is not an interesting limit from probabilistic point of view.

Wiener Process

Consider the following limit of the simple random walk

$\Delta x^2 \to 0$ as $\Delta t \to 0$

with

$$\Delta x = \sigma \Delta t; \quad p = \frac{1}{2}\left[1 + \frac{\mu\sqrt{\Delta t}}{\sigma}\right]; \quad q = \frac{1}{2}\left[1 - \frac{\mu\sqrt{\Delta t}}{\sigma}\right]$$

$\Rightarrow$

$$\langle S(t) \rangle \to \mu t$$

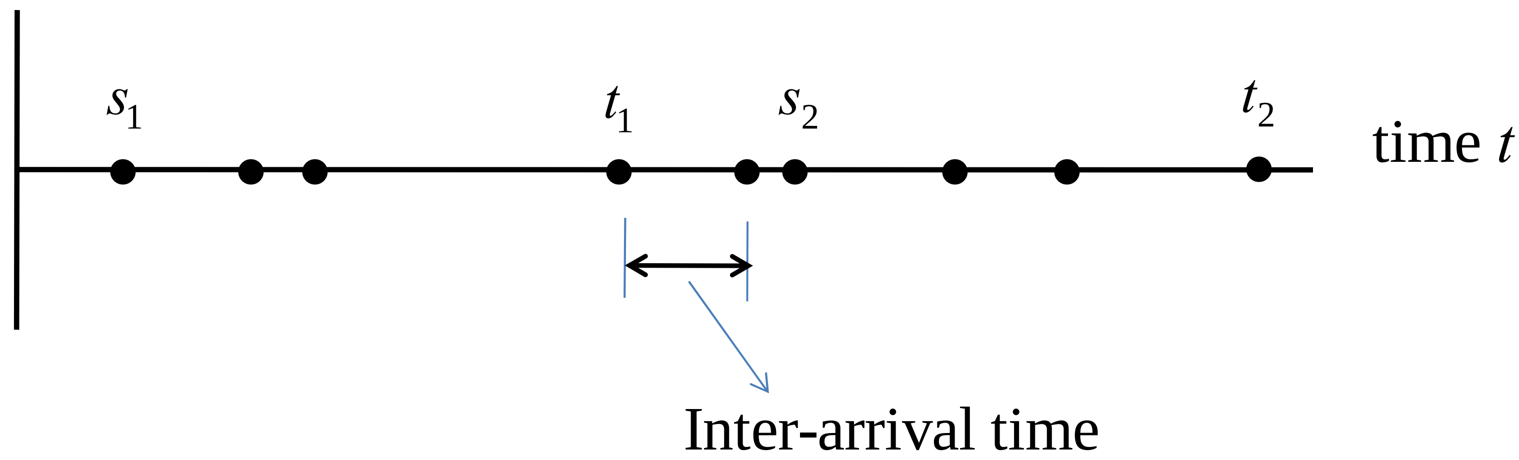
$$\mathrm{Var}\left[S(t)\right] \to \sigma^2 t$$

This is an interesting limit!

Remarks

- The resulting process is known as the Wiener process.
- This is a process with continuous state and continuous parameter.
- The process is a Gaussian process (central limit theorem).
- The process is nonstationary.
- If $\mu = 0$, the process is known as a Brownian motion process.

Random events and Poisson process



Let $N(t)$ be the number of events occuring randomly in the interval $(0,t]$.

If there exists probability functions

$P_N(n,t) = P\left[N(t) = n\right],\ P_{NN}(n_1,t_1;n_2,t_2) = P\left[N(t_1) = n_1 \cap N(t_2) = n_2\right]\cdots$

then we say that $N(t)$ is a counting process (discrete state, continuous parameter random process).

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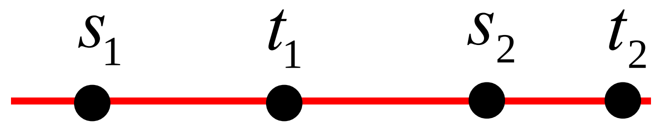
$N(t)$ is said to be a Poisson process with stationary increments if the following conditions are satisfied



(a) **Independent arrivals :**

That is, $P\left[N\left(t_1\right)-N\left(s_1\right)=n \mid N\left(t_2\right)-N\left(s_2\right)=m\right]=P\left[N\left(t_1\right)-N\left(s_1\right)=n\right]$

where $\left(s_1,t_1\right]\&\left(s_2,t_2\right]$ are mutually exclusive and $s_1<t_1 \;\&\; s_2<t_2$.

(b)**Stationary arrival rule**:

$$P\left[N\left(t+dt\right)-N(t)=1\right]=P\left[N\left(t+dt+h\right)-N(t+h)=1\right]=\lambda dt;\lambda>0.$$

(c) **Negligible probability for simultaneous arrivals**:

$$P\left[N\left(t+dt\right)-N(t)=1\right]=\lambda dt \;\&\; P\left[N\left(t+dt\right)-N(t)>1\right]=0.$$

Under these conditions it can be shown that

$$P\left[N(t)=k\right]=\frac{\left(\lambda t\right)^k}{k!}\exp\left(-\lambda t\right);k=0,1,2,\cdots,\infty.$$

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Proof

$$P_N[n,t+dt] = P_N[n,t]P_N[0,dt] + P_N[n-1,t]P_N[1,dt]$$

$$P_N[0,dt] = 1-\lambda dt$$

$$P_N[1,dt] = \lambda dt$$

$$\Rightarrow$$

$$P_N[n,t+dt] = P_N[n,t](1-\lambda dt) + P_N[n-1,t]\lambda dt$$

$$\Rightarrow \frac{P_N[n,t+dt]-P_N[n,t]}{dt} = -\lambda P_N[n,t] + P_N[n-1,t]\lambda = \lambda\left\{P_N[n-1,t]-P_N[n,t]\right\}$$

$$\Rightarrow \frac{d}{dt}P_N[n,t] + \lambda P_N[n,t] = \lambda P_N[n-1,t]$$

$$\Rightarrow P_N[n,t] = A_n \exp(-\lambda t) + \int_0^t \lambda \exp[-\lambda(t-\tau)]P_N[n-1,\tau]d\tau$$

This equation can be used to recursively evaluate $P_N[n,t]$ by varying n as $n = 0,1,2,\cdots$

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Thus with n=0, we have

$$P_N\left[0,t\right]=A_0\exp\left(-\lambda t\right)+\int_0^t\lambda\exp\left[-\lambda\left(t-\tau\right)\right]P_N\left[-1,\tau\right]d\tau$$

Clearly, $P_N\left[-1,\tau\right]=P\left[N\left(\tau\right)=-1\right]=0$

$$\Rightarrow P_N\left[0,t\right]=A_0\exp\left(-\lambda t\right)$$

We have $P_N\left[0,0\right]=P\left[N\left(0\right)=0\right]=1$ $\left(\because\right.$ counting begins after $\left.t=0\right)$

$$\Rightarrow 1=A_0\Rightarrow P_N\left[0,t\right]=\exp\left(-\lambda t\right)$$

Consider now $n=1$.

$$P_N\left[1,t\right]=A_1\exp\left(-\lambda t\right)+\int_0^t\lambda\exp\left[-\lambda\left(t-\tau\right)\right]P_N\left[0,\tau\right]d\tau$$

$$=A_1\exp\left(-\lambda t\right)+\int_0^t\lambda\exp\left[-\lambda\left(t-\tau\right)\right]\exp\left(-\lambda\tau\right)d\tau$$

$$=A_1\exp\left(-\lambda t\right)+\lambda t\exp\left(-\lambda t\right)$$

We have $P_N\left[1,0\right]=P\left[N\left(0\right)=1\right]=0$

$$\Rightarrow 0=A_1\Rightarrow P_N\left[1,t\right]=\lambda t\exp\left(-\lambda t\right)$$

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Repeating this process for $n = 2,3,\cdots$ we get

$$P_N[n,t] = \frac{(\lambda t)^n}{n!}\exp(-\lambda t); n = 0,1,2,\cdots,\infty$$

Remark

If the stationary arrival rule is relaxed, the above model can be modified to read as

$$P_N[n,t] = \frac{1}{n!}\left[\int_0^t \lambda(\tau)\,d\tau\right]^n \exp\left[-\int_0^t \lambda(\tau)\,d\tau\right]; n = 0,1,2,\cdots,\infty$$

Random pulses

Here we construct a random process by viewing it as a superposition of pulses arriving randomly in time.

$$X(t) = \sum_{k=1}^{N(t)} W_k\left(t, \tau_k\right)$$

$N\left(t\right) =$ counting process

$W_k\left(t, \tau_k\right) =$ a random pulse that commences at time τ_k.

Consider the subclass

$$X(t) = \sum_{k=1}^{N(t)} Y_k w\left(t, \tau_k\right)$$

$Y_k =$ iid sequence of rvs; independent of $\tau_k \forall k;$ indicate the intensity of the k-th event.

$w\left(t, \tau_k\right) =$ a deterministic pulse arriving at time τ_k; $w\left(t, \tau_k\right) = 0 \forall t < \tau_k$.

By imposing the condition $t < T$, we can write the above equation as

$$X(t) = \sum_{k=1}^{N(T)} Y_k w\left(t, \tau_k\right); \quad T > t$$

It can be shown that

$$m_X\left(t\right) = m_Y \int_0^t w\left(t, \tau\right) \lambda\left(\tau\right) d\tau,$$

$$C_{XX}\left(t_1, t_2\right) = E\left(Y^2\right) \int_0^{\min\left(t_1, t_2\right)} w\left(t_1, \tau\right) w\left(t_2, \tau\right) \lambda\left(\tau\right) d\tau, \&$$

$$\sigma_X^2\left(t\right) = E\left(Y^2\right) \int_0^t w^2\left(t, \tau\right) \lambda\left(\tau\right) d\tau$$

Example

Consider a random phenomenon E, which occurs as a Poisson process with constant arrival rate ν.

Let $t_1, t_2, \cdots, t_k$ be the times at which the event E occurs.

Let Z_i be the random variable representing the intensity measure of E occuring at the time instant t_i.

Let $Z_i, i = 1, 2, \cdots$ be an iid sequence with common PDF $P_Z(z)$.

Let $Z_{\max}(t)$ be the maximum value of Z_i observed over the time interval $(0, t)$.

Consider

$$P\left[Z_{\max} \le z \mid N(t)=k\right]=\left[P_Z(z)\right]^k$$

$\Rightarrow$

$$P_{Z_{\max}}(z)=\sum_{k=0}^{\infty} P\left[Z_{\max} \le z \mid N(t)=k\right] P\left[N(t)=k\right]$$

$$=\sum_{k=0}^{\infty}\left[P_Z(z)\right]^k \frac{(\nu t)^k}{k!}\exp(-\nu t)$$

$$=\exp\left[-\nu t\left(1-P_Z(z)\right)\right]$$

If $P_Z(z)=1-\exp\left[-\alpha\left(z-z_0\right)\right] \Rightarrow$

$$P_{Z_{\max}}(z)=\exp\left[-\nu t\left\{\exp\left[-\alpha\left(z-z_0\right)\right]\right\}\right]$$

This is the PDF of a Gumbel RV.

The above model has been used to model the maximum earthquake ground acceleration in the time interval 0 to t.

Differentiation and integration of random processes

Let $X(t)$ be a random process.

In formulating problems of mechanics

we need to differentiate random processes.

For example, if $X(t)$ is displacement,

we would be interested in velocity and acceleration.

Recall: for deterministic functions

$$\frac{dz}{dt} = \lim_{\Delta \to 0} \frac{z(t+\Delta) - z(t)}{\Delta}$$

By selecting a sequence of Δ-s, of the form $\{\Delta_{\mathrm{i}}\}_{i=1}^{n}$,

such that $\Delta_{\mathrm{i+1}} < \Delta_i$,

we obtain a sequence of numbers

$y_i = \dfrac{z(t+\Delta_i) - z(t)}{\Delta_i}$ and we seek to determine $\lim\limits_{i \to \infty} y_i$.

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When $X(t)$ is a random process, the sequence

$$Y_i = \frac{X(t+\Delta_i) - X(t)}{\Delta_i} \text{ , } i=1,2,\cdots$$

is a sequence of random variables.

What is meant by convergence of a sequence of random variables?

There are several valid modes of convergence of random variables. Consequently, the associated calculus also would be built based on a chosen mode of convergence of random variables.

Definition

A sequence of random variables $X_1, X_2, \cdots, X_n, \cdots$ is said to converge to the random variable X in the mean square sense if

$$\lim_{n\to\infty}\left\langle \left(X - X_i\right)^2 \right\rangle \to 0.$$

This is denoted by $\underset{n\to\infty}{\text{l.i.m.}} X_i \to X$.

The calculus based on this definition of convergence of rvs is called the mean square calculus.

This leads to the definition of mean square derivative and mean square integral.

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Consider

$$\left\langle X(t_1)\dot{X}(t_2)\right\rangle = \left\langle X(t_1)\,\underset{\Delta\to 0}{\text{l.i.m.}}\frac{X(t_2+\Delta)-X(t_2)}{\Delta}\right\rangle$$

$$= \lim_{\Delta\to 0}\frac{\left\langle X(t_1)X(t_2+\Delta)\right\rangle - \left\langle X(t_1)X(t_2)\right\rangle}{\Delta}$$

$$= \lim_{\Delta\to 0}\frac{R_{XX}(t_1,t_2+\Delta)-R_{XX}(t_1,t_2)}{\Delta} = \frac{\partial R_{XX}(t_1,t_2)}{\partial t_2}$$

$$R_{X\dot{X}}(t_1,t_2) = \frac{\partial R_{XX}(t_1,t_2)}{\partial t_2}$$

Similarly, it can be shown that

$$\left\langle \dot{X}(t_1)\dot{X}(t_2)\right\rangle = R_{\dot{X}\dot{X}}(t_1,t_2) = \frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}$$

& more generally,

$$\left\langle \left.\frac{d^n X}{dt^n}\right|_{t=t_1} \left.\frac{d^m X}{dt^m}\right|_{t=t_2}\right\rangle = \frac{\partial^{n+m} R_{XX}(t_1,t_2)}{\partial t_1^n \partial t_2^m}$$

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Remarks

(a) When we say that a random variable X exists in the mean square sense?

Answer: when $\sigma_X^2 < \infty$.

(b) Thus for $\dot{X}(t)$ to exist in the mean square sense, its variance must be finite. This means,

$$\lim_{t_1 \to t_2 = t} \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2} < \infty.$$

(c) If $X(t)$ and $Y(t)$ are jointly stationary, show that

$$\left\langle \frac{d^n X(t+\tau)}{dt^n} \frac{d^m Y(t)}{dt^m} \right\rangle = (-1)^m \frac{d^{n+m} R_{XY}(\tau)}{d\tau^{n+m}}$$

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Example : Show that for a zero mean, stationary random process, the process $X(t)$ and its derivative $\dot{X}(t)$ are uncorrelated.

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \exp(-i\omega\tau) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cos \omega\tau d\omega \quad \left[\because S_{XX}(-\omega) = S_{XX}(\omega)\right]$$

$$R_{\dot{X}X}(\tau) = -\frac{dR_{XX}(\tau)}{d\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega S_{XX}(\omega) \sin \omega\tau d\omega$$

$$\left\langle \dot{X}(t) X(t) \right\rangle = R_{\dot{X}X}(0) = 0$$

Example: Given $R_{XX}(t_1, t_2) = \lambda \min(t_1, t_2)$ determine $R_{\dot{X}\dot{X}}(t_1, t_2)$.

$$\frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} = \lambda \text{ if } t_2 < t_1$$

$$=0 \text{ if } t_1 > t_2$$

$$\Rightarrow \frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} = \lambda U(t_1 - t_2)$$

$$\Rightarrow \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2} = \lambda \delta(t_1 - t_2)$$

Example: Given $R_{XX}(t_1, t_2) = \lambda t_1 t_2 + \lambda \min(t_1, t_2)$

determine $R_{\dot{X}\dot{X}}(t_1, t_2)$.

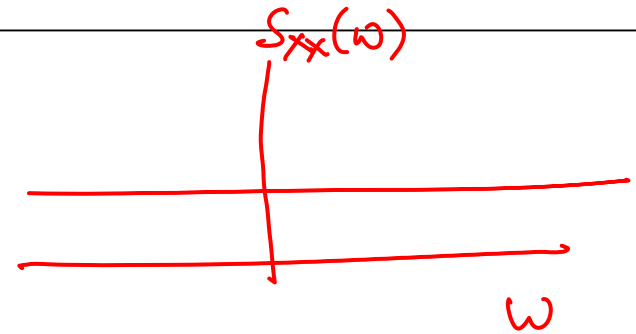
$$\frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} = \lambda t_1 \text{ if } t_1 < t_2$$

$$= \lambda t_1 + \lambda \text{ if } t_1 > t_2$$

$$\Rightarrow \frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} = \lambda t_1 + \lambda U(t_1 - t_2)$$

$$\Rightarrow \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2} = \lambda + \lambda \delta(t_1 - t_2)$$

White noise



$$R(\tau) = I\delta(\tau)$$

$$S(\omega) = I$$

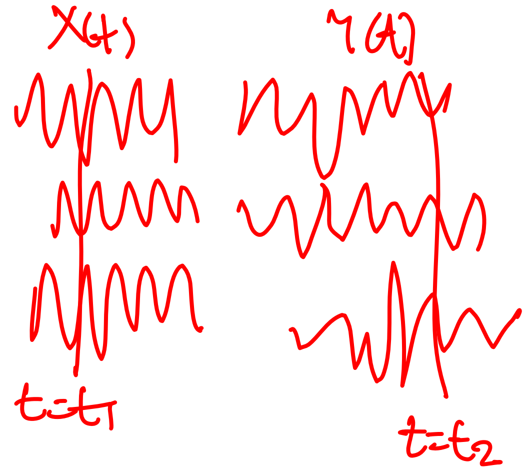
Area under psd (=variance) $\to \infty$.

The process is physically unrealizable.

Analogous to a concentrated load in mechanics and an impulse in dynamics.

Two random processes

Consider $X(t)$ & $Y(t)$ to be two random processes



$\cancel{P_{XX}}$ $P_{XY}(x, y; t_1, t_2)$

$$= P\left[X(t_1) \le x \cap Y(t_2) \le y\right]$$

$$p_{XY}(x, y; t_1, t_2) = \frac{\partial^2 P_{XY}(x, y; t_1, t_2)}{\partial x \partial y}$$

$$\underline{\underline{C_{XY}}}(t_1, t_2) = \left\langle [X(t_1) - m_X(t_1)][Y(t_2) - m_Y(t_2)] \right\rangle$$

$$= \iint_{-\infty}^{\infty} p_{XY}(x, y; t_1, t_2)\{x - m_X(t_1)\}\{y - m_Y(t_2)\}\,dx\,dy$$

Joint Stationary $\quad C_{XY}(t_1, t_2) = C_{XY}(t_1 - t_2)$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} C_{XY}(\tau) \exp(i\omega\tau)\,d\tau$$

$$C_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega) \exp(-i\omega\tau)\,d\omega$$