Stochastic Structural Dynamics

Lecture-7

Random processes-2

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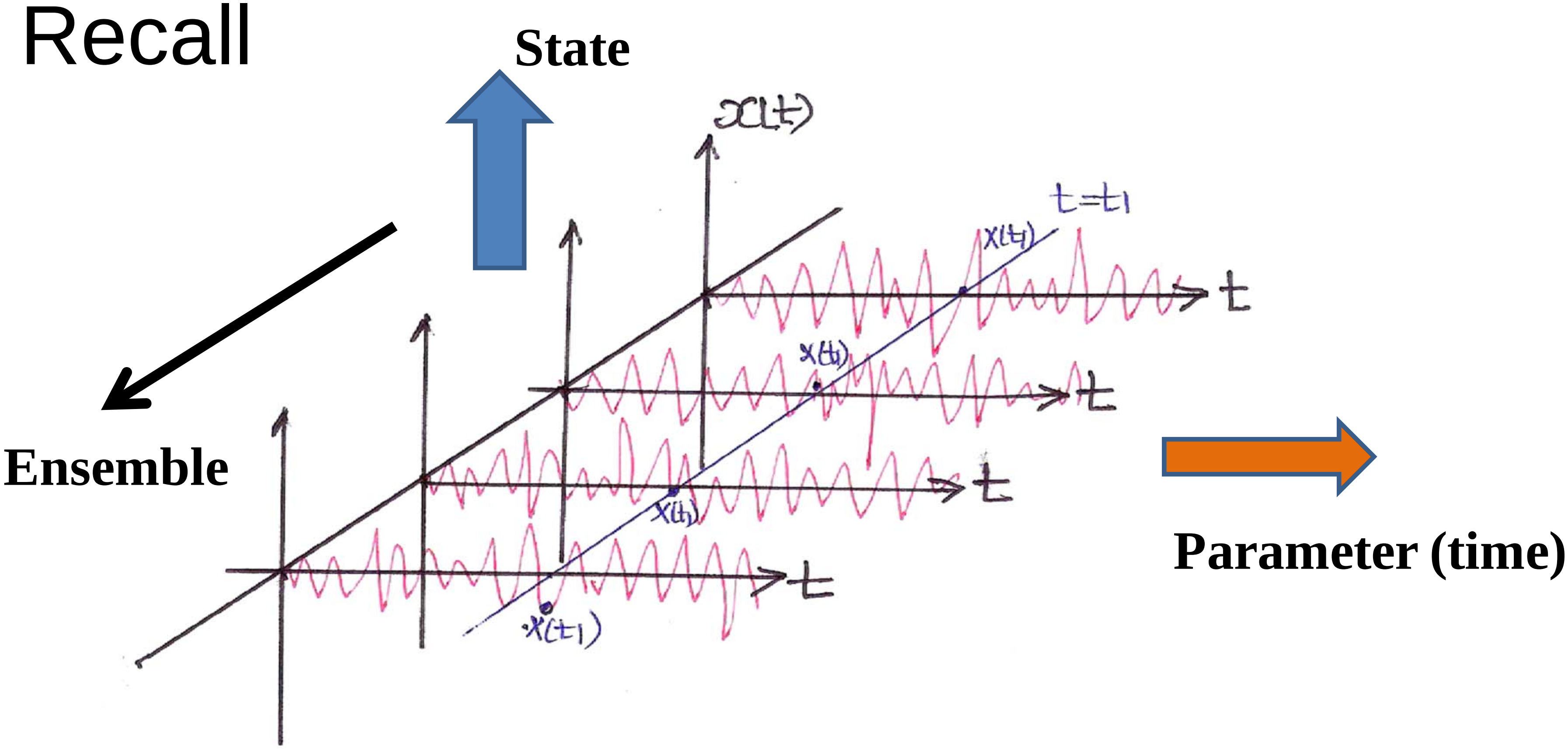
Recall



Multi-dimensional PDF, pdf
Mean, covariance,…
Stationarity: SSS, WSS
Eergodicty: Temporal and ensemble average

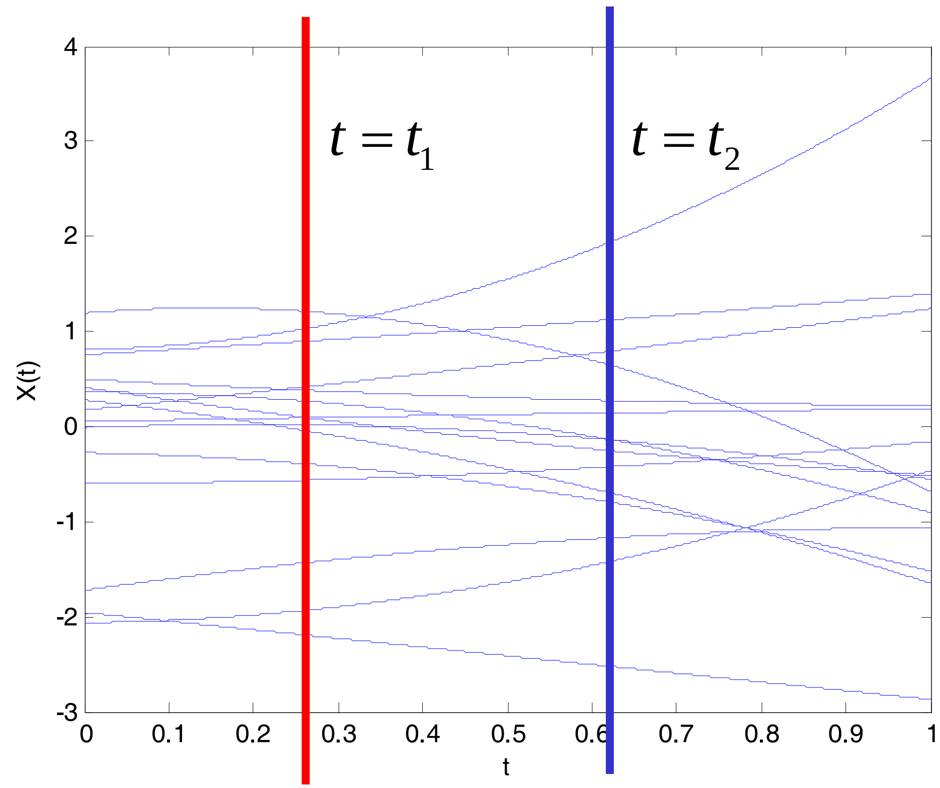
$$X(t) = a + bt + ct^2$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\langle X(t) \rangle = \langle a \rangle + \langle b \rangle t + \langle c \rangle t^2 = 0$$

$$\langle X(t_1) X(t_2) \rangle = a^2 + b^2 t_1 t_2 + c^2 t_1^2 t_2^2$$

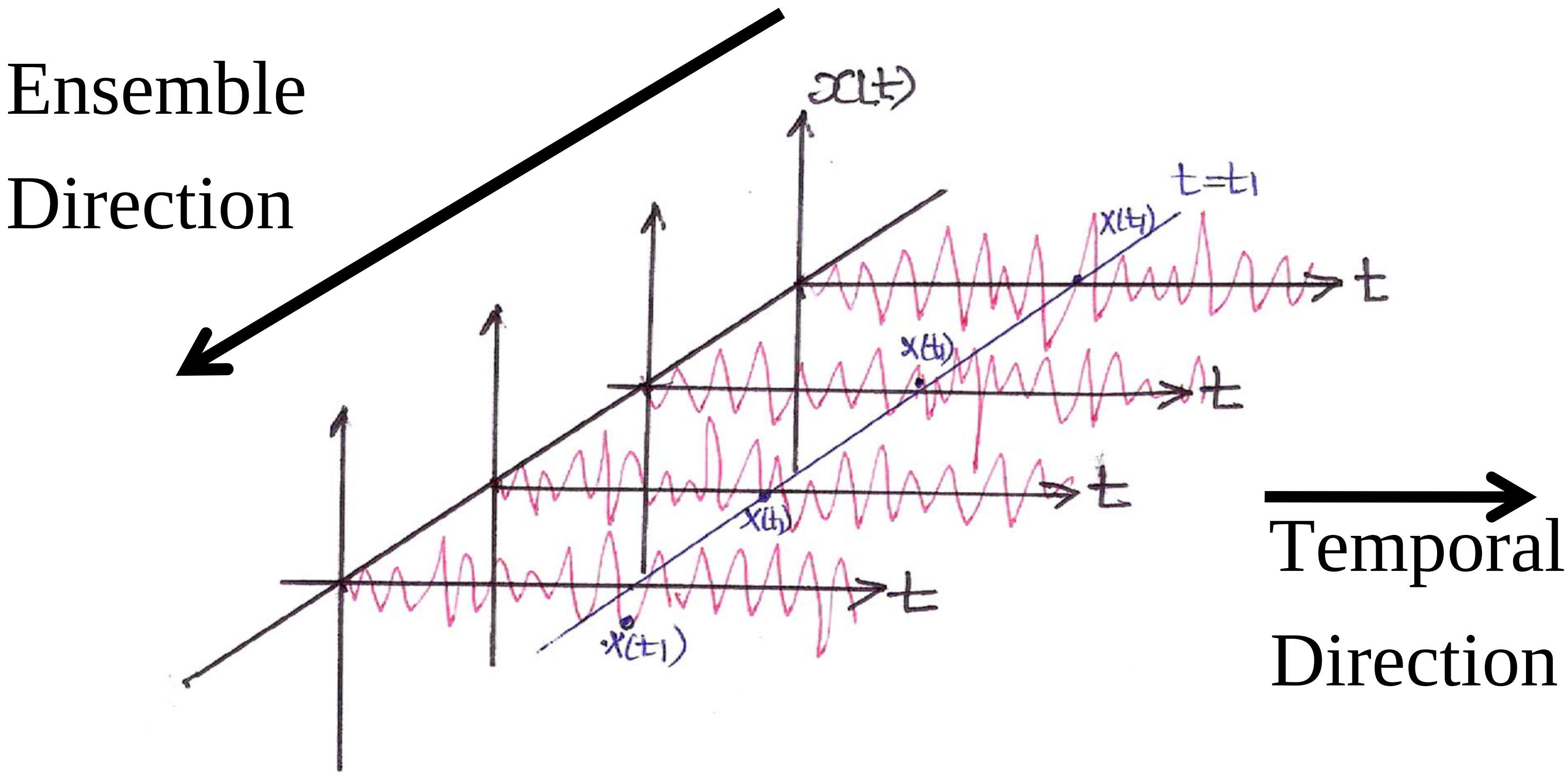
$$X(t) \sim N\left(0, a^2 + b^2 t^2 + c^2 t^4\right)$$



Ergodicty of a random process

Basic notion

Equivalence of temporal and ensemble averages



Let $x(t)$ be a sample realization of the random process $X(t)$. We define the time average of a given function of $X(t)$, $g[X(t)]$ by

$$\mathrm{T_{av}}\{g[X(t)]\} = \frac{1}{T}\int_0^T g[x(t)]\mathrm{d}t$$

If $X(t)$ is an ergodic random process, then $< g[X(t)] >= \mathrm{T_{av}}\{g[X(t)]\}$.

Definitions

- **Ergodicity in mean** $X(t)$ is ergodic in mean if

$$\mathrm{T}_{\mathrm{av}}\{X(t)\} = \frac{1}{T}\int_0^T x(t)\mathrm{d}t =< X(t) >$$

- **Ergodicity in the mean square** $X(t)$ is ergodic in meansquare if

$$\mathrm{T}_{\mathrm{av}}\{X^2(t)\} = \frac{1}{T}\int_0^T x^2(t)\mathrm{d}t =< X^2(t) >$$

- **Ergodicity in autocorrelation** $X(t)$ is said to be ergodic in auto-correlation if

$$\mathrm{T}_{\mathrm{av}}\{X(t)X(t1+\tau)\} = \frac{1}{T}\int_0^T x(t)x(t+\tau)\mathrm{d}t =< X(t)X(t+\tau) >= R_X(\tau)$$

Remaraks

1. The above list of definitions of ergodicity are not exhaustive: several other similar definitions can be constructed by considering other descriptors of the random process.

2. Ergodic processes are necessarily stationary in nature; a stationary random process need not be ergodic.

3. Physically, ergodicity means that a sufficiently long record of a stationary random process contains all the statistical information about the random phenomenon.

Ergodicity in mean

Let *X(t)* be a stationary random process with specified joint pdf structure

$$\eta_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$\Rightarrow \eta_T$ is a random variable

$$E\left[\eta_T\right] = \frac{1}{2T}\int_{-T}^{T} E[X(t)]\,dt = E\left[x(t)\right] = \eta$$

$$\sigma_{\eta_T}^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E\left[\left\{X(t_1)-\eta\right\}\left\{X(t_2)-\eta\right\}\right]dt_1 dt_2$$

$$= \frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)[R(\tau)-\eta^2]\,d\tau$$

Ergodicity in mean

X(t) is said to be ergodic in mean iff

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)dt=E\left[x(t)\right]=\eta$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)\left[R(\tau)-\eta^{2}\right]d\tau\to 0$$

Ergodicity in first order PDF

$$P_X(x,t) = P_X(x) = P\left[X(t) \le x\right]$$

$$Define$$

$$y(t) = 1 \; if \;\; X(t) \le x$$

$$y(t) = 0 \; if \;\; X(t) > x$$

$$\Rightarrow E\left[y(t)\right] = 1 \times P\left[X(t) \le x\right] + 0 \times P\left[X(t) > x\right] = P_X(x)$$

X(t) is said to be ergodic in first order PDF if *y(t)* is ergodic in mean

Ergodicity in autocorrelation

$$\text{Define}$$
$$\phi(t) = X(t)X(t+\tau)$$
$$E\left[\phi(t)\right] = E\left[X(t)X(t+\tau)\right] = R_{XX}(\tau)$$

X(t) is said to be ergodic in autocorrelation if *ϕ(t)* is ergodic in mean

- Criteria for ergodicity in other properties could be developed on similar lines

- The above criteria are applicable if description of the random process is available.

- The notion of ergodicity plays a crucial role in relating observed data to mathematical models of uncertainties

Frequency domain representation of functions of time

Let *x(t)* be a deterministic function of time

Time signals

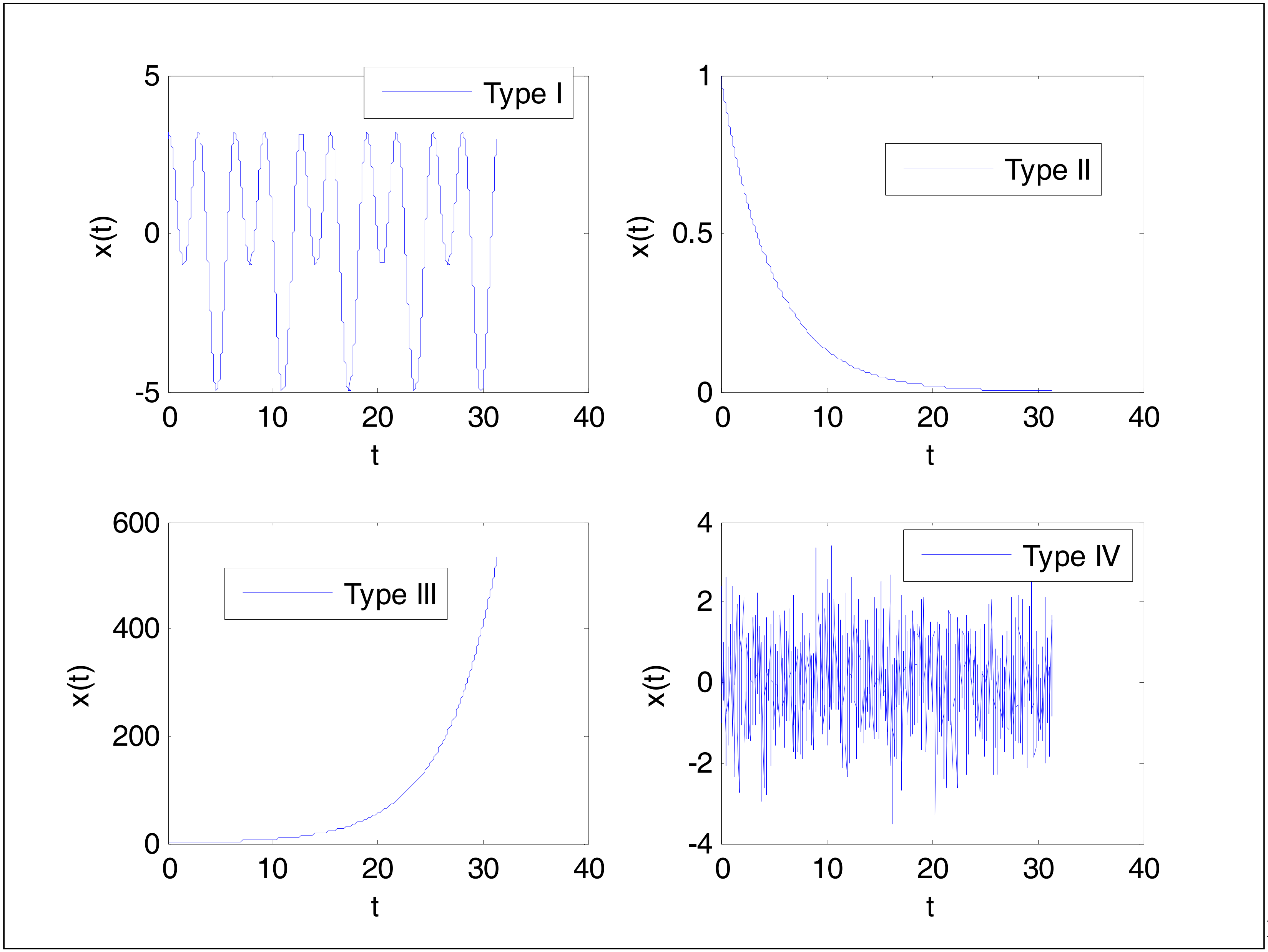
Type I : Periodic signals (well behaved) $x(t \pm nT) = x(t)$

Type II : Aperiodic signals $\lim_{t\to\infty} |x(t)| \to 0$

Type III : Aperiodic signals $\lim_{t\to\infty} |x(t)| \to \infty$

Type IV : Aperiodic signals $\lim_{t\to\infty} |x(t)|$ neither goes to zero nor becomes unbounded.

A classification of time signals



Remarks

Realizations of stationary random process belong to Type IV signals.

Type III signals

No hope of any frequency domain representation.

Type I functions

Periodic signals $y(t) = y(t \pm nT)$

Period: the smallest value of T for which the above condition is valid.

$$y(t) = P\sin\lambda t = P\sin(\lambda t + 2\pi) = P\sin\lambda\left(t + \frac{2\pi}{\lambda}\right) \Rightarrow T = \frac{2\pi}{\lambda}$$

$$y(t) = P\cos\lambda t = P\cos(\lambda t + 2\pi) = P\cos\lambda\left(t + \frac{2\pi}{\lambda}\right) \Rightarrow T = \frac{2\pi}{\lambda}$$

$$y(t) = P\cos\lambda t + Q\sin\lambda t = P\cos(\lambda t + 2\pi) + Q\sin(\lambda t)$$
$$= P\cos\lambda\left(t + \frac{2\pi}{\lambda}\right) + Q\sin\lambda\left(t + \frac{2\pi}{\lambda}\right) \Rightarrow T = \frac{2\pi}{\lambda}$$

$$y(t) = P\cos 2\lambda t = P\cos(2\lambda t + 2\pi)$$
$$= P\cos 2\lambda\left(t + \frac{2\pi}{2\lambda}\right) \Rightarrow T = \frac{\pi}{\lambda}$$

$$y(t) = P\cos\lambda t + Q\cos 2\lambda t$$
$$= P\cos(\lambda t + 2\pi) + Q\cos(2\lambda t + 2\pi) \Rightarrow T = \frac{2\pi}{\lambda}$$

$$y(t) = \sum_{n=1}^{N} a_n \cos(\frac{2\pi n}{T} t) + b_n \sin(\frac{2\pi n}{T} t) \Rightarrow$$

Y(t) is periodic with period=T

According to Fourier's theorem, under general conditions, a periodic function *y(t)* can be represented by

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n}{T} t) + b_n \sin(\frac{2\pi n}{T} t)$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos\left(\frac{2\pi n t}{T}\right) dt \quad \& \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin\left(\frac{2\pi n t}{T}\right) dt; n = 1, 2, \cdots, \infty$$

Recall

$$\cos\theta = \frac{1}{2}\left[\exp(i\theta) + \exp(-i\theta)\right] \quad \& \quad \sin\theta = \frac{1}{2i}\left[\exp(i\theta) - \exp(-i\theta)\right] \Rightarrow$$

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\exp\left(i\frac{2\pi nt}{T}\right) + \exp\left(-i\frac{2\pi nt}{T}\right)}{2} \right) + b_n \left(\frac{\exp\left(i\frac{2\pi nt}{T}\right) - \exp\left(-i\frac{2\pi nt}{T}\right)}{2i} \right)$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \exp\left(i\frac{2\pi nt}{T}\right)(a_n - ib_n) + \exp\left(-i\frac{2\pi nt}{T}\right)(a_n + ib_n)$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n \exp\left(i\frac{2\pi nt}{T}\right)$$

$$\alpha_n = \frac{a_n - ib_n}{2}; \quad \text{with} \quad a_{-n} = a_n; \quad b_{-n} = -b_n$$

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} y(t) \exp\left(-i\frac{2\pi nt}{T}\right) dt$$

sine, cosine, amplitude and phase spectra

$x(t)$ is periodic with period T

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos\omega_n t + b_n \sin\omega_n t\}; \quad \omega_n = \frac{2\pi n}{T}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\omega_n t dt \;\; \& \;\; b_n = \frac{2}{T}\int_0^T x(t)\sin\omega_n t dt$$

- The plots of a_n and b_n as a function of ω_n are called, respectively, as the Fourier cosine and sine spectra.
- The plot of $\sqrt{a_n^2 + b_n^2}$ as a function of ω_n is called the Fourier amplitude spectrum.
- The plot of $\tan^{-1}\left(\frac{b_n}{a_n}\right)$ as a function of ω_n is called the Fourier phase spectrum.

Energy and power of a signal

If $x(t)$ is a displacement function, $x^2(t)$ is a quantity that is proportional to potential energy. Similarly, if $x(t)$ is a velocity function, $x^2(t)$ is a quantity that is proportional to kinetic energy.

We call $\lim_{s\to\infty}\int_0^s x^2(t)\,dt$ as the total energy in the signal.

We call $\frac{1}{T}\int_0^T x^2(t)\,dt$ as the energy per cycle (power) in the signal.

Total energy and power Discrete power spectrum

Total energy: $\lim_{s\to\infty}\int_0^s x^2(t)\,dt \to \infty \Rightarrow$

Total energy is not an useful concept.

Energy per cycle= $\dfrac{1}{\mathrm{T}}\int_0^T x^2(t)\,dt$ makes sense.

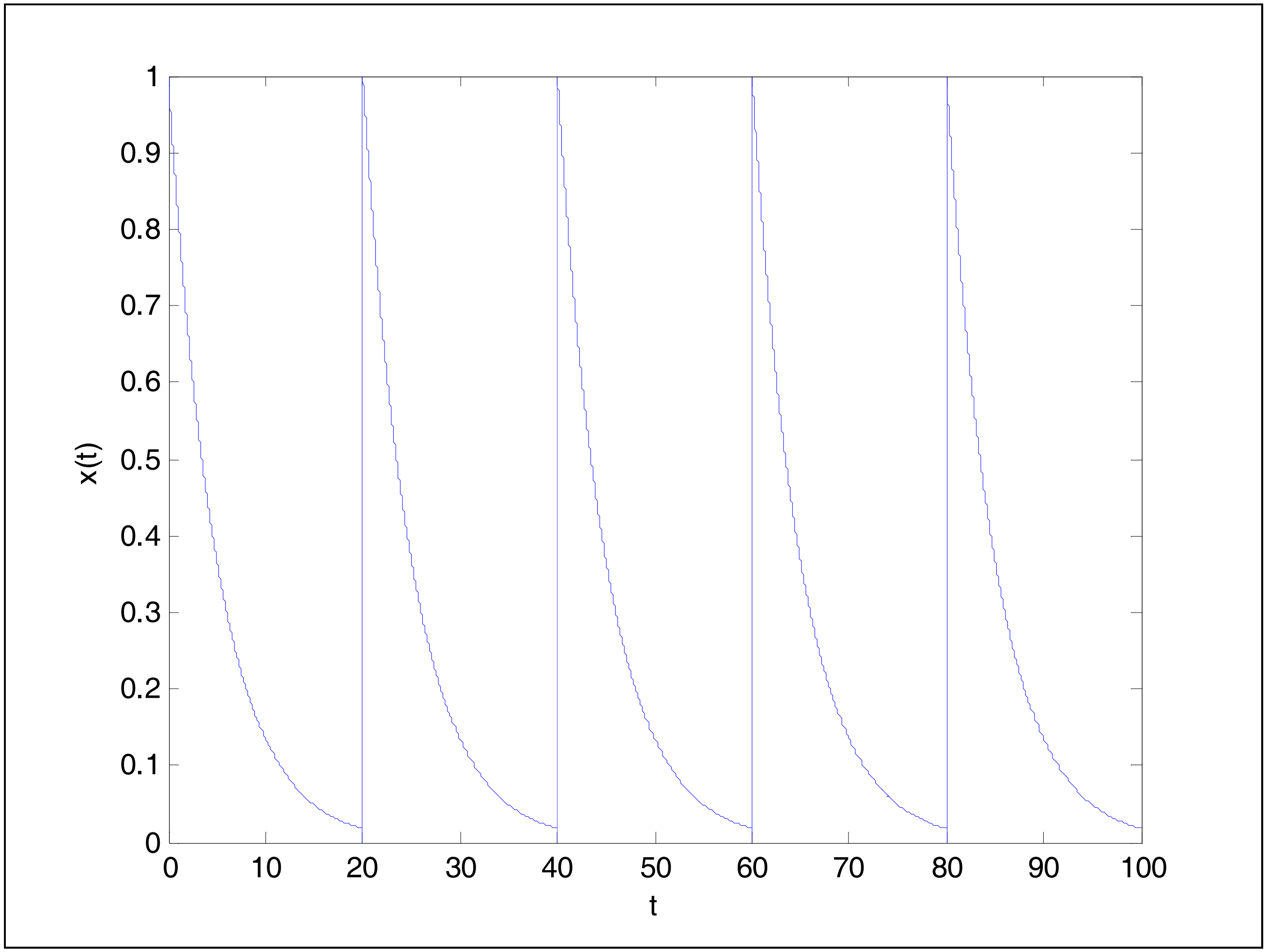
The plot of $\dfrac{a_n^2+b_n^2}{2}$ as a function of ω_n is called the discrete power spectrum.

Discrete power spectrum is an useful concept for Type I signals.

Type II signals

$$x_T(t) = x(t) \text{ for } 0 < t < T$$
$$x_T(t + nT) = x(t) \text{ for } n = 1, 2, \cdots, \infty$$

$$x(t) = \exp(-0.2t)$$



$$x_T(t) = x(t) \text{ for } 0 < t < T$$

$$x_T(t + nT) = x(t) \text{ for } n = 1, 2, \cdots, \infty$$

$x_T(t)$ belongs to Type I of time functions.

$\Rightarrow$ $x_T(t)$ admits a Fourier series representation.

Clearly, $\lim_{T \to \infty} x_T(t) \to x(t)$.

Question: What happens to Fourier series based description of $x_T(t)$ as $T \to \infty$?

$$x_T(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp\left(\frac{i2\pi nt}{T}\right)$$

$$\alpha_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\exp\left(-\frac{i2\pi nt}{T}\right)dt$$

$$x_T(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(s)\exp\left(-\frac{i2\pi ns}{T}\right)ds\right]\exp\left(\frac{i2\pi nt}{T}\right)$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(s)\exp\left(-i2\pi nf_0 s\right)ds\right]\exp\left(i2\pi nf_0 t\right)$$

$$x_T(t)=\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x_T(s)\exp(-i2\pi nf_0 s)\,ds\right]\exp(i2\pi nf_0 t)$$

$$f_0=\frac{1}{T}\Rightarrow f_n=\frac{n}{T}\ \&\ f_{n+1}=\frac{n+1}{T}$$

$$\Rightarrow f_{n+1}-f_n=\frac{1}{T}=\Delta f_n=\Delta f$$

$$x_T(t)=\sum_{n=-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}}x_T(s)\exp(-i2\pi nf_0 s)\,ds\right]\exp(i2\pi nf_0 t)\Delta f_n$$

$$=\sum_{n=-\infty}^{\infty}X(f_n)\exp(i2\pi nf_0 t)\Delta f_n$$

$$\lim_{\Delta f_n\to\infty}x_T(t)\to x(t)=\int_{-\infty}^{\infty}X(f)\exp(i2\pi ft)\,df$$

Definition: Fourier Transform pair

$x(t)$ is aperiodic; $\lim_{t\to\infty} \left| x(t) \right| \to 0$

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp[i2\pi ft] df$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp[-i2\pi ft] dt$$

$$\text{Power} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t) dt \ \to 0 \text{ and hence not useful.}$$

$$\text{Total energy} = \lim_{s\to\infty} \int_0^s x^2(t) dt \to \text{ could be an useful quantity.}$$

Energy spectrum is an useful concept

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(i\omega t) d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-i\omega t) dt$$

x(t) and X(ω) are said to form a Fourier transform pair

Parseval theorem

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} X(f)\exp(i2\pi ft)df\right]dt$$

$$= \int_{-\infty}^{\infty} X(f)\left[\int_{-\infty}^{\infty} x(t)\exp(-i2\pi ft)dt\right]df$$

$$= \int_{-\infty}^{\infty} X(f)X^*(f)df$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} \left|X(f)\right|^2 df$$

Type III time functions

$$\lim_{t\to\infty}\left|x(t)\right| \to \infty$$

No hope of any frequency domain representations

Type IV

Define $x_T\left(t\right) = x(t)$ for $0 < t \le T$ &

$= 0$ for $t > T$

$$X_T\left(\omega\right) = \int_{-\infty}^{\infty} x_T\left(t\right)\exp\left(-i\omega t\right)dt$$

$$x_T\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T\left(\omega\right)\exp\left(i\omega t\right)d\omega$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left|X_T\left(f\right)\right|^2 df = \text{ Total power}$$

$$\Rightarrow h\left(f\right) = \lim_{T\to\infty}\frac{1}{T}\left|X_T\left(f\right)\right|^2 = \text{power spectral density function.}$$

$\Rightarrow h\left(f\right)df =$ contribution to the total power made by the frequency components in the range $\left(f, f+df\right)$.

Type V: *x(t)* is a stationary random process

Let $X(t)$ be a zero mean stationary random process.

Samples of $X(t)$ belong to Type IV time histories.

$\Rightarrow$ For each sample the power spectral density function can be defined.

Definition:

Power spectral density funciton of $X(t)$

$$S_{XX}(f) = \lim_{T \to \infty} \frac{1}{T} \left\langle \left| X_T(f) \right|^2 \right\rangle$$

$$
\begin{aligned}
S_{XX}(\omega) &= \lim_{T\to\infty} \frac{1}{T} \left\langle X_T(\omega) X_T^*(\omega) \right\rangle \\
&= \lim_{T\to\infty} \frac{1}{T} \left\langle \int_0^T X(t) \exp(-i\omega t)\, dt \int_0^T X(t) \exp(i\omega t)\, dt \right\rangle \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T \left\langle X(t_1) X(t_2) \right\rangle \exp\left[i\omega (t_2 - t_1) \right] dt_1 dt_2 \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T R_{XX}(t_2 - t_1) \exp\left[i\omega (t_2 - t_1) \right] dt_1 dt_2 \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \left[T - |\tau| \right] R_{XX}(\tau) \exp(i\omega\tau)\, d\tau
\end{aligned}
$$

If we restrict our attention to only those $R_{XX}(\tau)$ which satisfy the condition

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}|\tau|R_{XX}(\tau)\exp(i\omega\tau)\,d\tau\to 0,$$

we get the relations

$$S_{XX}(\omega)=\int_{-\infty}^{\infty}R_{XX}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{XX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega$$

Remarks

(1) $R_{XX}(\tau) = \left\langle X(t)X(t+\tau)\right\rangle = \left\langle X(t)X(t-\tau)\right\rangle = R_{XX}(-\tau)$

(2) $$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(i\omega\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)(\cos\omega\tau + i\sin\omega\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos\omega\tau\,d\tau \quad \left\{\because R_{XX}(\tau) = R_{XX}(-\tau)\right\}$$

$$= 2\int_{0}^{\infty} R_{XX}(\tau)\cos\omega\tau\,d\tau$$

Remarks

$$(3) R_{XX}(0) = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega.$$

$\Rightarrow$ Area under the PSD function is the variance of the process.

$(4) S_{XX}(\omega) d\omega =$ contribution to the total average power (variance) made by frequency components in the range $(\omega, \omega + d\omega)$.

$\Rightarrow S_{XX}(\omega) \geq 0$

$$(5) S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-i\omega\tau) d\tau \qquad (\text{Substitute } s = \text{-}\tau)$$

$$= \int_{-\infty}^{\infty} R_{XX}(-s)\exp(i\omega s) ds$$

$$= \int_{-\infty}^{\infty} R_{XX}(s)\exp(i\omega s) ds = S_{XX}(\omega)$$

Remarks

(6) $$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S_{XX}(\omega)\cos\omega\tau\,d\omega$$

(7) Physical PSD function (defined only for $\omega \geq 0$)

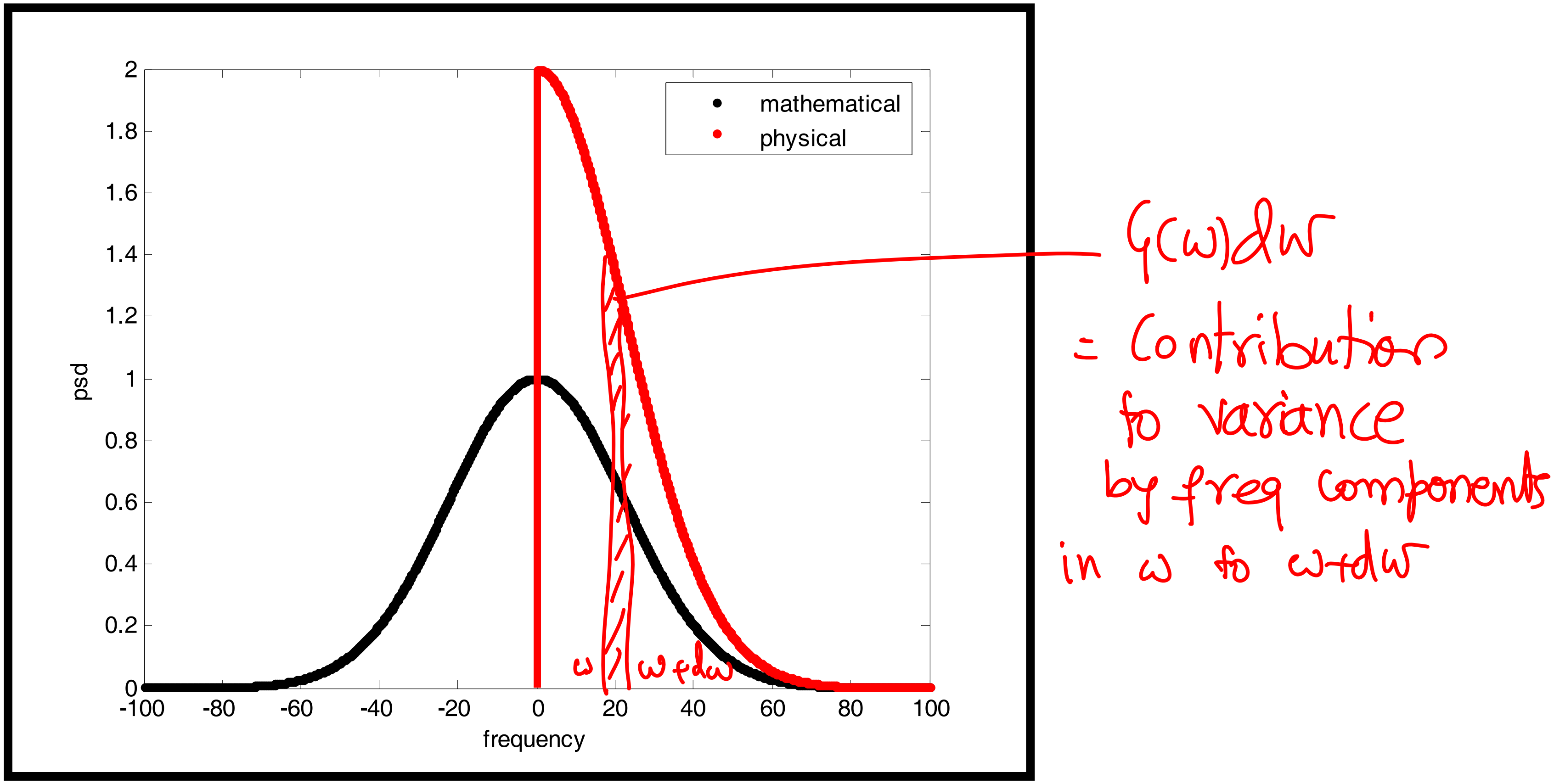
$G_{XX}(\omega) = 2S_{XX}(\omega)$ for $\omega \geq 0$

=0 for $\omega < 0$

Area under $G_{XX}(\omega)$ would still be the variance of the process.

PSD

$$\frac{S_{XX}(\omega)}{\sigma_X^2} \text{ has properties similar to a pdf}$$



Units of PSD: $\dfrac{\left[\text{Units of } X(t)\right]^2}{\text{frequency}}$

Ex: $X(t)$ is displacement

Units of PSD : $\dfrac{\text{m}^2}{\text{Hz}}$ or $\dfrac{\text{m}^2}{(\text{rad/s})}$

Similarly, if X(t) is acceleration

Units of PSD: $\dfrac{(\text{m/s}^2)^2}{\text{Hz}}$ or $\dfrac{(\text{m/s}^2)^2}{(\text{rad/s})}$

Remarks

(8) Wiener-Khinchine relations

$$S_{XX}(\omega) = 2\int_0^\infty R_{XX}(\tau)\cos\omega\tau\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^\infty S_{XX}(\omega)\cos\omega\tau\, d\omega$$

$\Rightarrow$

$$G_{XX}(\omega) = 4\int_0^\infty R_{XX}(\tau)\cos\omega\tau\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_0^\infty G_{XX}(\omega)\cos\omega\tau\, d\omega$$

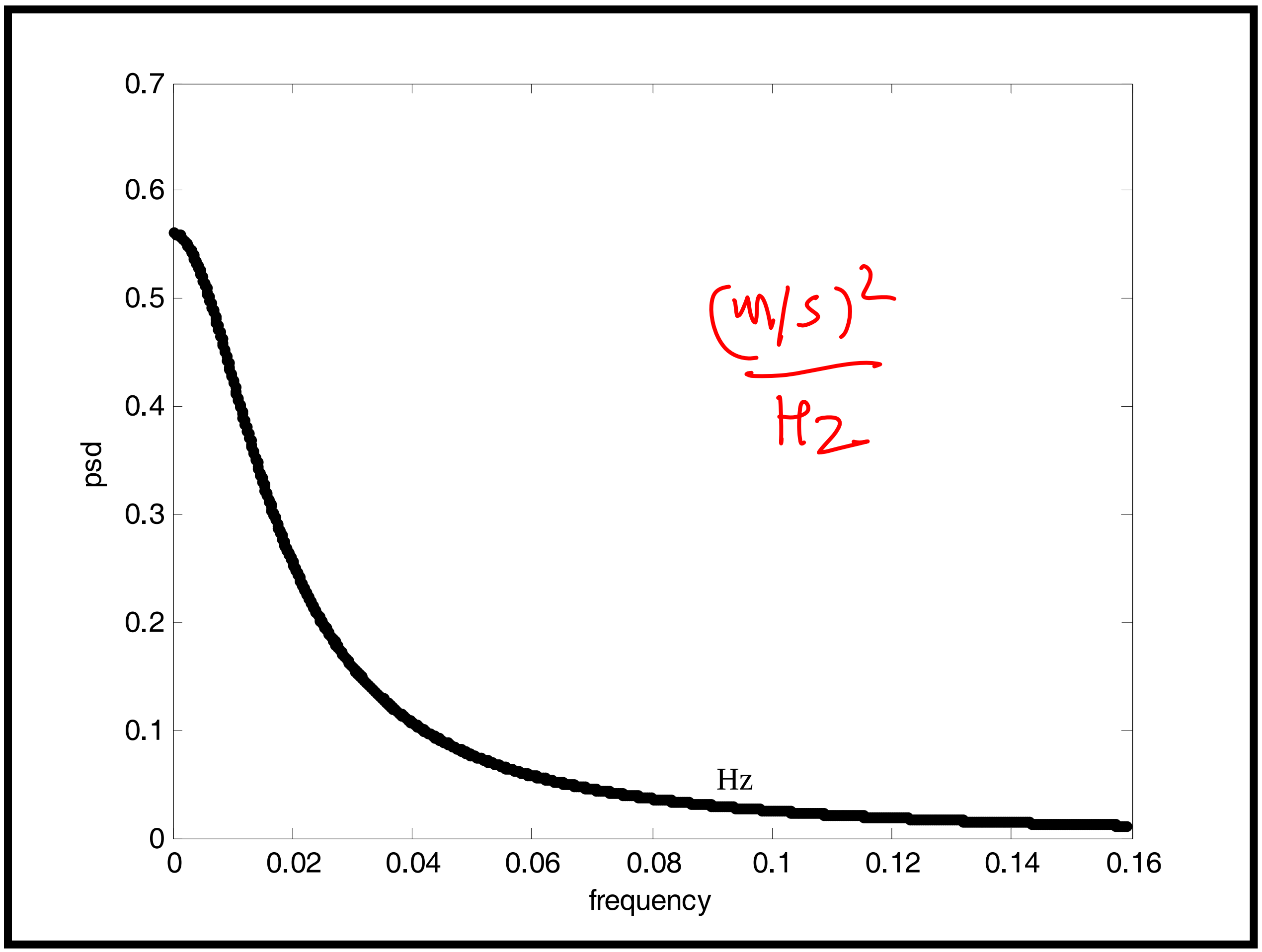
A few examples of covariance and psd function pairs

$$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\exp(j\omega\tau)\,d\omega \qquad S(\omega)=\int_{-\infty}^{\infty} R(\tau)\exp(-j\omega\tau)\,d\omega$$

$\delta(\tau)$	1
$\exp(j\beta\tau)$	$2\pi\delta(\omega-\beta)$
1	$2\pi\delta(\omega)$
$\cos\beta\tau$	$\pi\delta(\omega-\beta)+\pi\delta(\omega+\beta)$
$\exp(-\alpha\lvert\tau\rvert)$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$
$\exp(-\alpha\tau^2)$	$\sqrt{\dfrac{\pi}{\alpha}}\exp\left(-\dfrac{\omega^2}{4\alpha}\right)$
$\exp(-\alpha\lvert\tau\rvert)\cos\beta\tau$	$\dfrac{\alpha}{\alpha^2+(\omega-\beta)^2}+\dfrac{\alpha}{\alpha^2+(\omega+\beta)^2}$
$2\exp(-\alpha\tau^2)\cos\beta\tau$	$\sqrt{\dfrac{\pi}{\alpha}}\left\{\exp\left(-\dfrac{(\omega-\beta)^2}{4\alpha}\right)+\exp\left(-\dfrac{(\omega+\beta)^2}{4\alpha}\right)+\right\}$
$\dfrac{\sin\sigma\tau}{\pi\tau}$	$\begin{cases}1 & \lvert\omega\rvert<\sigma\\ 0 & \lvert\omega\rvert>\sigma\end{cases}$

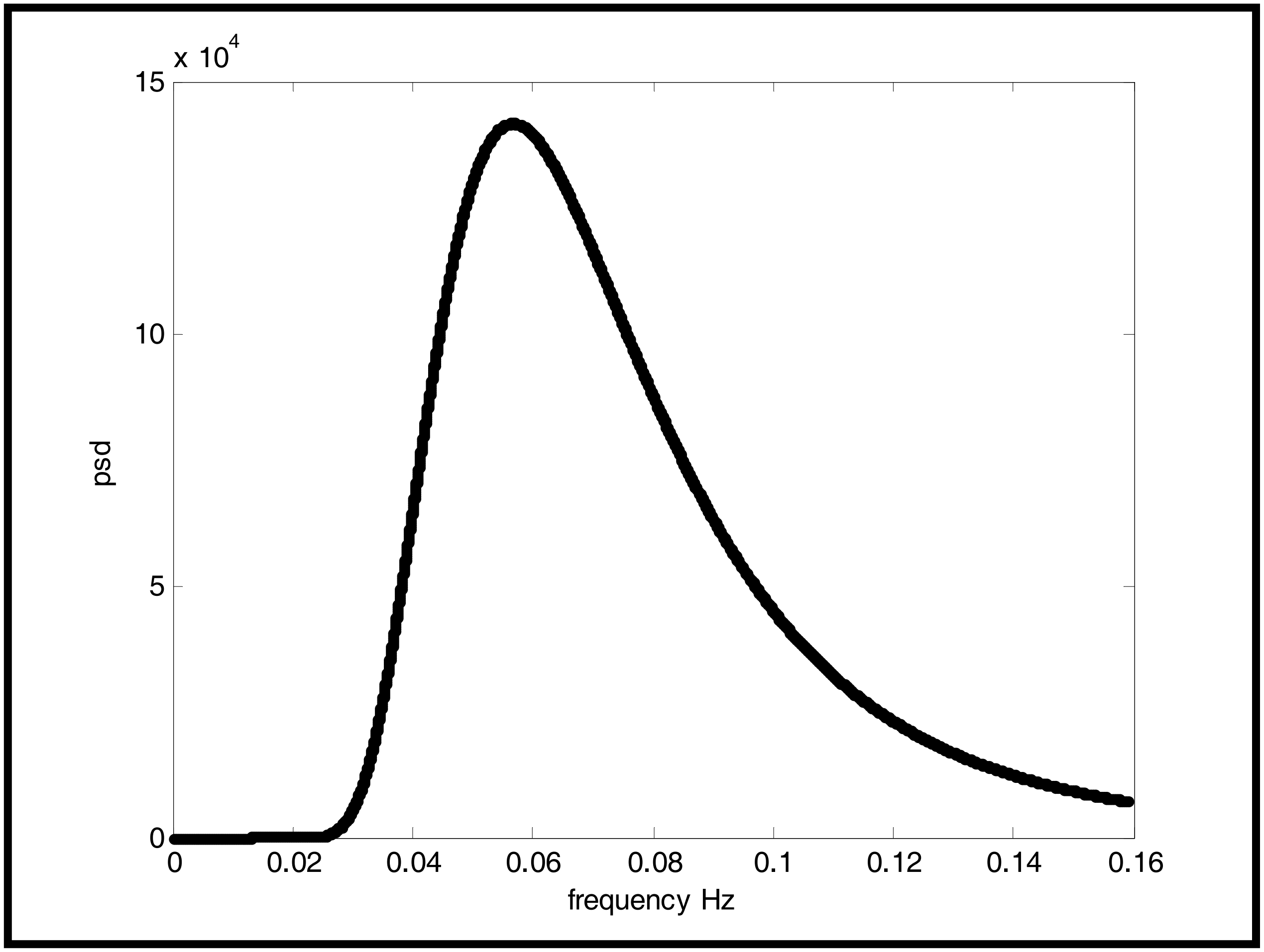
Typical psd function of wind velocity

$$G(\omega) = c_0 \frac{1}{\left[2 + \left(\frac{873.6\omega}{u_0}\right)^2\right]^{\frac{5}{6}}}$$

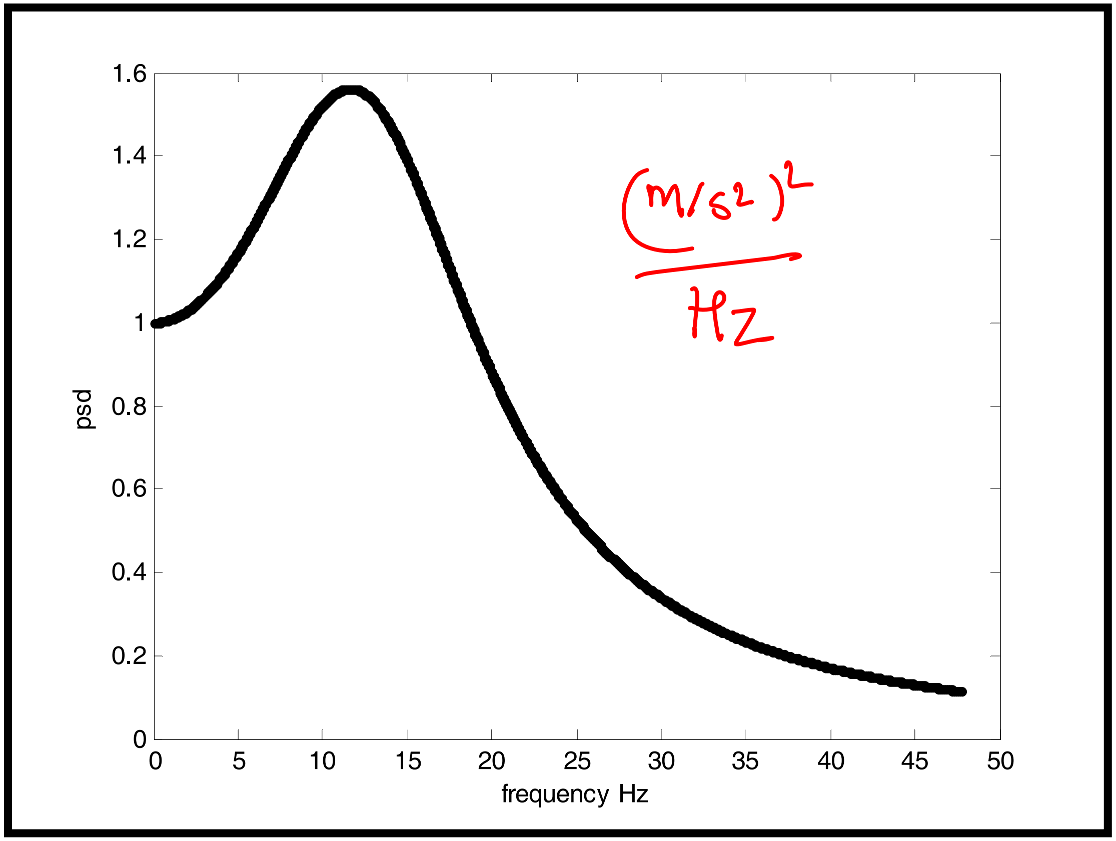


Typical psd function of waves

$$G(\omega) = c_0 \frac{1}{\omega^5} \exp\left(-\frac{c_1}{\omega^4}\right)$$



Typical psd function of earthquake ground acceleration



$$G(\omega) = c_0 \frac{1 + \left(\frac{\omega}{\omega_g}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_g}\right)^2\right]^2 + 4\eta_g^2 \left(\frac{\omega}{\omega_g}\right)^2}$$

Evolutionary random process

Consider a random process $X(t)$ defined as

$X(t) = V_1(t)$ if $0 < t < t_1$

& $X(t) = V_2(t)$ if $t > t_1$.

Let $V_1(t)$ & $V_2(t)$ be zero mean, stationary random processes

with psd functions $\left\{S_{VVi}(\omega)\right\}_{i=1}^{2}$.

$\Rightarrow$ We can write

$S_{XX}(\omega,t) = S_{VV_1}(\omega)$ *if* $0 < t < t_1$

$S_{XX}(\omega,t) = S_{VV_2}(\omega)$ *if* $t > t_1$

This notion can be generalized to define nonstationary random processes with time dependent psd functions.

Such processes are called as evolutionary random processes.