Stochastic Structural Dynamics

Lecture-40

Problem solving session-4

Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

Problem 40

A sdof system driven by a filtered

Gaussian excitation is governed by the equations

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f(t); x(0) = 0; \dot{x}(0) = \dot{x}_0$$

$$\dot{f} + \alpha f = \xi(t)$$

Here $\xi(t)$ is zero mean Gaussian white noise process such that $\langle \xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)$

- Set up the equations for time evolution of first two order moments using Markov process approach.
- Consider the response in the steady state and evaluate the response moments

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f(t); x(0) = 0; \dot{x}(0) = \dot{x}_0$$

$$\dot{f} + \alpha f = \xi(t)$$

$$\langle \xi(t) \rangle = 0; \langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_2 - t_1)$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \\ f \end{Bmatrix}$$

$$\Rightarrow$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2\eta\omega x_2 - \omega^2 x_1 + x_3$$

$$\dot{x}_3 = -\alpha x_3 + \xi(t)$$

$$dx_1 = x_2 dt$$

$$dx_2 = \left(-2\eta\omega x_2 - \omega^2 x_1 + x_3\right) dt$$

$$dx_3 = -\alpha x_3 + dB(t)$$

$$\langle dB(t)\rangle = 0 \,\&\, \langle dB(t)\, dB(t+\tau)\rangle = 2D\delta(\tau)$$

Recall

$$dX(t) = f\left[X(t), t\right] dt + G\left[X(t), t\right] dB(t); t \geq 0; X(0) = X_0$$

$$X(t), f \sim n\times 1; dB(t) \sim m\times 1; G \sim n\times m$$

$$\frac{d}{dt}\left\langle h\left[X(t), t\right]\right\rangle =$$

$$\left\langle \frac{\partial h}{\partial t}\right\rangle + \sum_{j=1}^{n}\left\langle f_j(X,t)\frac{\partial h}{\partial X_j}\right\rangle + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \left(GDG^t\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\right\rangle$$

$$\frac{d}{dt}\left\langle h\left[X(t),t\right]\right\rangle =$$

$$\left\langle \frac{\partial h}{\partial t}\right\rangle + \sum_{j=1}^{n}\left\langle f_j(X,t)\frac{\partial h}{\partial X_j}\right\rangle + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \left(GDG^t\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\right\rangle$$

$$\frac{d}{dt}\left\langle h\left[X(t),t\right]\right\rangle = \left\langle X_2\frac{\partial h}{\partial X_1}\right\rangle + \left\langle \left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right)\frac{\partial h}{\partial X_2}\right\rangle$$

$$-\left\langle \alpha X_3\frac{\partial h}{\partial X_3}\right\rangle + \frac{D}{2}\left\langle \frac{\partial^2 h}{\partial X_3^2}\right\rangle$$

$$\frac{d}{dt}\langle X_1 \rangle = \langle X_2 \rangle$$

Steady state

$\langle x_2 \rangle = 0$

$$\frac{d}{dt}\langle X_2 \rangle = \langle -2\eta\omega X_2 - \omega^2 X_1 + X_3 \rangle$$

RH = 0

$$\frac{d}{dt}\langle X_3 \rangle = -\langle \alpha X_3 \rangle$$

$\langle \alpha X_3 \rangle = 0$

$$\frac{d}{dt}\left\langle X_1^2 \right\rangle = 2\left\langle X_1 X_2 \right\rangle$$

$$\frac{d}{dt}\left\langle X_2^2 \right\rangle = \left\langle \left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right) 2X_2 \right\rangle$$

$$\frac{d}{dt}\left\langle X_3^2 \right\rangle = -\left\langle \alpha X_3 2X_3 \right\rangle + D$$

$$\frac{d}{dt}\left\langle X_1 X_2 \right\rangle = \left\langle X_2^2 \right\rangle + \left\langle \left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right) X_1 \right\rangle$$

$$\frac{d}{dt}\left\langle X_1 X_3 \right\rangle = \left\langle X_2 X_3 \right\rangle - \alpha \left\langle X_3 X_1 \right\rangle$$

$$\frac{d}{dt}\left\langle X_2 X_3 \right\rangle = \left\langle \left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right) X_3 \right\rangle - \alpha \left\langle X_3 X_2 \right\rangle$$

Initial conditions

Assume that $x_0, \dot{x}_0$ & $f(0) = f_0$ are all deterministic.

$\Rightarrow$

$$\langle \mathrm{X}_1(0) \rangle = x_0$$

$$\langle \mathrm{X}_2(0) \rangle = \dot{x}_0$$

$$\langle \mathrm{X}_3(0) \rangle = f_0$$

$$\langle X_1^2 \rangle, \langle X_2^2 \rangle, \langle X_3^2 \rangle, \langle X_1 X_2 \rangle, \langle X_1 X_3 \rangle, \langle X_2 X_3 \rangle = 0 \,@\, t = 0$$

Steady state response analysis

steady state $\Rightarrow$

$$\frac{d}{dt}\left\langle h\left[X(t),t\right]\right\rangle = 0 \Rightarrow$$

$$\left\langle X_2 \frac{\partial h}{\partial X_1}\right\rangle + \left\langle \left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right)\frac{\partial h}{\partial X_2}\right\rangle - \left\langle \alpha X_3 \frac{\partial h}{\partial X_3}\right\rangle + \frac{D}{2}\left\langle \frac{\partial^2 h}{\partial X_3^2}\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_1\right\rangle = \left\langle X_2\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_2\right\rangle = \left\langle -2\eta\omega X_2 - \omega^2 X_1 + X_3\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_3\right\rangle = -\left\langle \alpha X_3\right\rangle = 0$$

$$\Rightarrow \left\langle X_1\right\rangle = \left\langle X_2\right\rangle = \left\langle X_3\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_1^2\right\rangle = 2\left\langle X_1 X_2\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_2^2\right\rangle = \left\langle\left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right)2X_2\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_3^2\right\rangle = -\left\langle\alpha X_3 2X_3\right\rangle + D = 0$$

$$\frac{d}{dt}\left\langle X_1 X_2\right\rangle = \left\langle X_2^2\right\rangle + \left\langle\left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right)X_1\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_1 X_3\right\rangle = \left\langle X_2 X_3\right\rangle - \alpha\left\langle X_3 X_1\right\rangle = 0$$

$$\frac{d}{dt}\left\langle X_2 X_3\right\rangle = \left\langle\left(-2\eta\omega X_2 - \omega^2 X_1 + X_3\right)X_3\right\rangle - \alpha\left\langle X_3 X_2\right\rangle = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4\eta\omega & 0 & -2\omega^2 & 0 & 2 \\ 0 & 0 & -4\alpha & 0 & 0 & 0 \\ -\omega^2 & 1 & 0 & -2\eta\omega & 1 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1 \\ 0 & 0 & 1 & 0 & -\omega^2 & -2\eta\omega-\alpha \end{bmatrix} \begin{Bmatrix} \langle X_1^2 \rangle \\ \langle X_2^2 \rangle \\ \langle X_3^2 \rangle \\ \langle X_1 X_2 \rangle \\ \langle X_1 X_3 \rangle \\ \langle X_2 X_3 \rangle \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0 \\ D \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow$$

$$\begin{Bmatrix} \langle X_1^2 \rangle \\ \langle X_2^2 \rangle \\ \langle X_3^2 \rangle \\ \langle X_1 X_2 \rangle \\ \langle X_1 X_3 \rangle \\ \langle X_2 X_3 \rangle \end{Bmatrix} = -\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4\eta\omega & 0 & -2\omega^2 & 0 & 2 \\ 0 & 0 & -4\alpha & 0 & 0 & 0 \\ -\omega^2 & 1 & 0 & -2\eta\omega & 1 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1 \\ 0 & 0 & 1 & 0 & -\omega^2 & -2\eta\omega-\alpha \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ D \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Problem 41

In a study on reliability analysis of a cracked plate it has become necessary to simulate a vector of six non-Gaussian random variables. The specification of these random variables is limited to the description of the 1st order pdf-s and the matrix of correlation coefficients. Develop a simulation procedure based on the Nataf transformation to simulate 5000 samples of the random variables. Estimate the 1st order PDF-s from the simulated sample and perform the Kolmogorov-Smirnov test to verify if the simulations have been performed satisfactorily.

The distribution of the basic random variables are as follows:

$$X_1 \sim N(60,10); X_2 \sim LN(1,0.2); X_3 \sim LN(2,0.1);$$

$$X_4 \sim EX(1); (X_5, X_6) \sim N_2(-33.0, 0.47, 3.5, 0.3, -0.9)$$

$$\rho = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & -0.835 \\ 0 & 0 & 0 & 0 & -0.835 & 1.0 \end{bmatrix}$$

Partially specified non-Gaussian RVs
Nataf's transformation

Let X_1 and X_2 be two random variables such that

- X_1 and X_2 are not completely specified
- Knowledge on X_1 and X_2 is limited to first order pdfs and the covariance matrix.

Question: How to transform X to standard normal space?

Let

$$P_{X_1}(X_1) = \Phi(U_1)$$

$$P_{X_2}(X_2) = \Phi\left(U_2\right)$$

with $U_1 \sim N(0,1), U_2 \sim N(0,1) \,\&\, \left\langle U_1 U_2 \right\rangle = \rho_{12}^*$

$$\Rightarrow p_1(x_1)\frac{dx_1}{du_1} = \phi(u_1);\ \frac{dx_1}{du_2} = 0$$

$$p_2(x_2)\frac{dx_2}{du_2} = \phi(u_2);\frac{dx_2}{du_1} = 0$$

$$J = \begin{vmatrix} \dfrac{\phi(u_1)}{p_1(x_1)} & 0 \\ 0 & \dfrac{\phi(u_2)}{p_2(x_2)} \end{vmatrix} = \frac{\phi(u_1)\phi(u_2)}{p_1(x_1)\,p_2(x_2)}$$

$$p_{X_1X_2}(x_1,x_2)=\frac{p_{U_1U_2}(u_1,u_2)}{\phi(u_1)\phi(u_2)}p_1(x_1)p_2(x_2)$$

$$@u_1=\Phi^{-1}\left[P_{X_1}(x_1)\right],u_2=\Phi^{-1}\left[P_{X_2}(x_2)\right]$$

$$=\frac{\phi_2\left\{\Phi^{-1}\left[P_{X_1}(x_1)\right],u_2=\Phi^{-1}\left[P_{X_2}(x_2)\right]\right\}}{\phi\left\{\Phi^{-1}\left[P_{X_1}(x_1)\right]\right\}\phi\left\{\Phi^{-1}\left[P_{X_2}(x_2)\right]\right\}}p_1(x_1)p_2(x_2)$$

$$\rho_{12}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_1)(x_2-\mu_2)p_{X_1X_2}(x_1,x_2)dx_1dx_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_1)(x_2-\mu_2)\frac{\phi_2\left\{\Phi^{-1}\left[P_{X_1}(x_1)\right],u_2=\Phi^{-1}\left[P_{X_2}(x_2)\right]\right\}}{\phi\left\{\Phi^{-1}\left[P_{X_1}(x_1)\right]\right\}\phi\left\{\Phi^{-1}\left[P_{X_2}(x_2)\right]\right\}}$$

$$p_1(x_1)p_2(x_2)dx_1dx_2$$

Substitute

$$P_{X_1}\left(x_1\right)=\Phi\left(z_1\right)\,\&\,P_{X_2}\left(x_2\right)=\Phi\left(z_1\right)$$

$$\Rightarrow dx_1 dx_2\, p_{\mathrm{X}_1}\left(x_1\right) p_{\mathrm{X}_1}\left(x_1\right)=\phi\left(z_1\right)\phi\left(z_1\right)dz_1 dz_2$$

$$\Rightarrow \rho_{12}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(P_{X_1}^{-1}\left\{\Phi\left(z_1\right)\right\}-\mu_1)(P_{X_1}^{-1}\left\{\Phi\left(z_1\right)\right\}-\mu_2)\phi_2\left(z_1,z_2,\rho_{12}^{*}\right)dz_1 dz_2$$

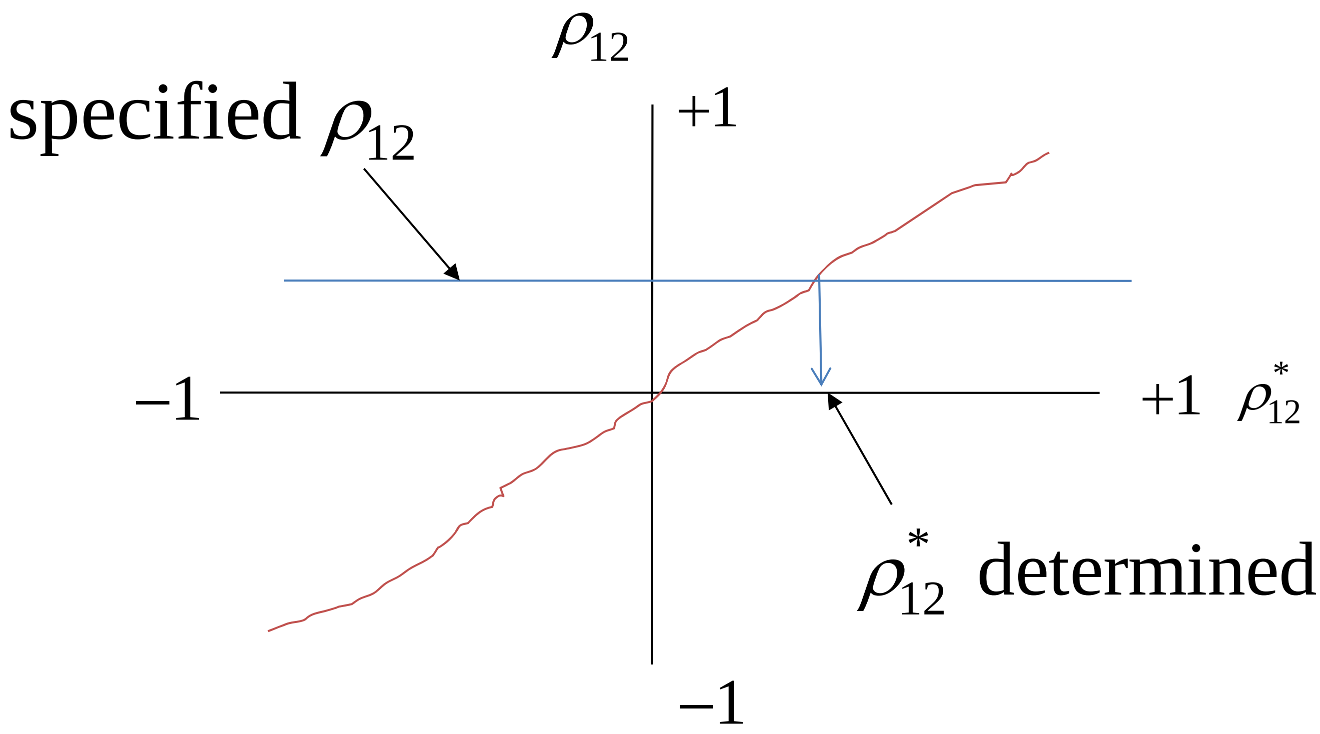
Strategy for the determination of the unknown ρ_{12}^{*}

$$\rho_{12} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (P_{X_1}^{-1}\left\{\Phi\left(z_1\right)\right\} - \mu_1)(P_{X_2}^{-1}\left\{\Phi\left(z_2\right)\right\} - \mu_2)\phi_2\left(z_1, z_2, \rho_{12}^{*}\right)dz_1 dz_2$$

(1) Divide the range [-1,1] of ρ_{12}^{*} into L divisions.

(2) For each value of $\left\{\rho_{12}^{*i}\right\}_{i=1}^{L}$ solve the above equation (numerically) and obtain the corresponding values of $\left\{\rho_{12}^{i}\right\}_{i=1}^{L}$. Note that $-1 \le \rho_{12}^{i} \le 1 \forall i = 1, 2, \cdots, L.$

(3) Interpolate $\left\{\rho_{12}^{i}\right\}_{i=1}^{L}$ to obtain the value of ρ_{12}^{*} for which the target value of ρ_{12} is realized.



ρ_{12}
specified ρ_{12}
+1
−1
+1 ρ_{12}^*
ρ_{12}^* determined
−1

Steps for simulation of 2 - dimensional Nataf random variables

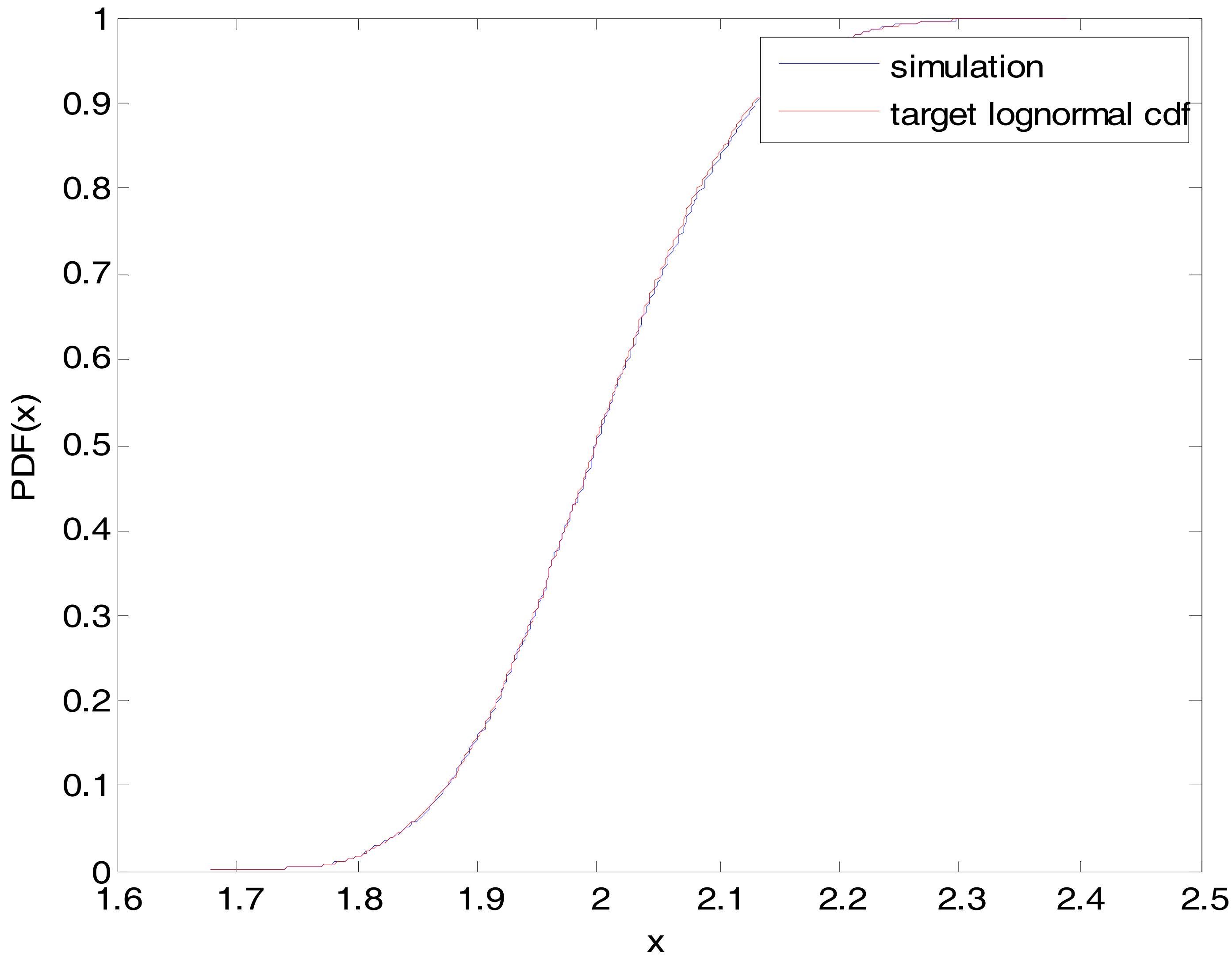
Step 1 solve for ρ_{12}^* by solving

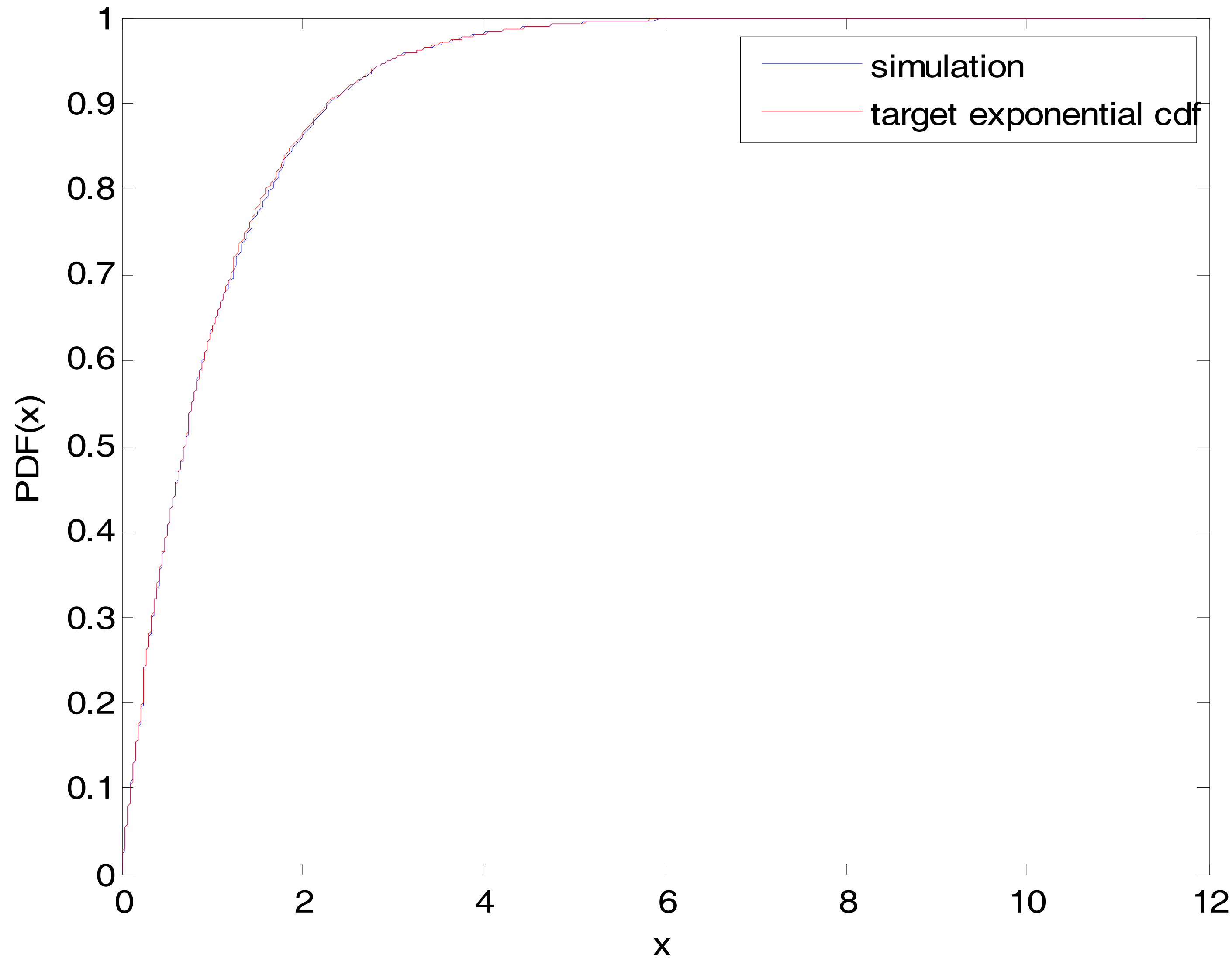
$$\rho_{12} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (P_{X_1}^{-1}\left\{\Phi\left(z_1\right)\right\} - \mu_1)(P_{X_2}^{-1}\left\{\Phi\left(z_2\right)\right\} - \mu_2)\phi_2\left(z_1, z_2, \rho_{12}^*\right)dz_1 dz_2$$

Step 2 Simulate $Z \sim N\left(0, \rho^*\right)$.

Step 3 Simulate X_1 and X_2 using

$$X_i = P_X^{-1}\left\{\Phi\left(U_i\right)\right\}; i = 1, 2$$


simulation
target lognormal cdf
PDF(x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.6
1.7
1.8
1.9
2
2.1
2.2
2.3
2.4
2.5



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
PDF(x)
0
2
4
6
8
10
12
x
simulation
target exponential cdf

Rho_equivalent_gaussian

rho_t =

1.0000	0	0	0	0	0
0	1.0000	0	0	0	0
0	0	1.0000	0	0	0
0	0	0	1.0000	0	0
0	0	0	0	1.0000	-0.8833
0	0	0	0	-0.8833	1.0000

Msim=

59.8787	1.0019	1.9990	1.0044	5.1723e-015	3.4961

Stdsim=

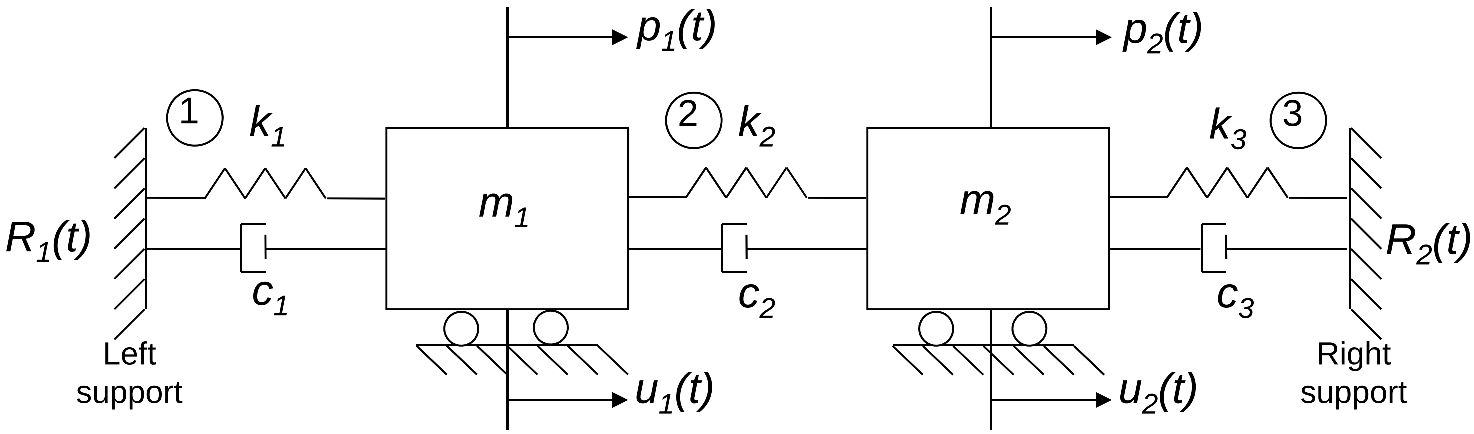
9.9726	0.1986	0.0998	1.0104	2.5838e-015	0.2982

Rhosim=

1.0000	-0.0154	0.0093	-0.0135	-0.0013	0.0126
-0.0154	1.0000	0.0206	0.0025	0.0014	-0.0068
0.0093	0.0206	1.0000	-0.0016	0.0018	-0.0058
-0.0135	0.0025	-0.0016	1.0000	-0.0180	0.0116
-0.0013	0.0014	0.0018	-0.0180	1.0000	-0.8345
0.0126	-0.0068	-0.0058	0.0116	-0.8345	1.0000

Problem 42

A two dof system with cubic and hysteretic nonlinear stiffness characteristics is shown in the figure.



Springs k_1 and k_2 have cubic force displacement characteristics and spring k_3 is an inelastic spring modeled using Bouc's approach.

The system is taken to be governed by the equations

$$m_1\ddot{u}_1 + c_1\dot{u}_1 + c_2\left(\dot{u}_1 - \dot{u}_2\right) + k_1 u_1 + \alpha_1 u_1^3 + k_2\left(u_1 - u_2\right)$$

$$+\alpha_2\left(u_1 - u_2\right)^3 = p_1\left(t\right) + w_1\left(t\right)$$

$$m_2\ddot{u}_2 + c_2\left(\dot{u}_2 - \dot{u}_1\right) + c_3\dot{u}_2 + k_2\left(u_2 - u_1\right) + \alpha_2\left(u_2 - u_1\right)^3$$

$$+k_3\bar{\lambda}u_2 + k_3\bar{z}\left(1 - \bar{\lambda}\right) = p_2\left(t\right) + w_2\left(t\right)$$

$$\dot{\bar{z}} = -\gamma\,|\dot{u}_2|\,|\bar{z}|\,|\bar{z}|^{\bar{n}-1} - \beta\dot{u}_2\,|\bar{z}|^{\bar{n}} + A\dot{u}_2 + w_3\left(t\right)$$

$$\dot{p}_1 + \bar{\alpha}_1 p_1 = w_4\left(t\right)$$

$$\dot{p}_2 + \bar{\alpha}_2 p_2 = w_5\left(t\right)$$

$\left\{w_i(t)\right\}_{i=1}^{5}$ are given to be a set of independent white noise processes with $\left\langle w_i(t)\, w_i(t+\tau)\right\rangle = \sigma_i^2 \delta(\tau)$.

The various system paramters are as follows:

$m_1 = 1.0\text{kg}, m_2 = 1.5\text{kg}, k_1 = 0.1\text{kN / m}, k_2 = 0.2\text{kN/m},$

$k_3 = 0.15\text{kN/m}, \eta_1 = \eta_2 = 0.05,$

$\alpha_1 = 2, \alpha_2 = 4, \bar{\lambda} = 0.05, \gamma = 0.5, \beta = 0.5, A = 1, \bar{n} = 3,$

$\bar{\alpha}_1 = 10, \bar{\alpha}_2 = 20, T = 15.6s, t_0 = 2s, \Delta = 0.0042s$

$\sigma_1 = 0.01\text{N}, \sigma_2 = 0.02\text{N}, \sigma_3 = 0.01\text{N}, \sigma_4 = 10.0\text{N}$ and

$\sigma_5 = 20.0\text{N}$

Using 1.5 order strong Taylor's scheme, develop a procedure to simulate samples of the sytem response. Hence estimate the response moments (up to second order) and the first order pdf-s.

Recall

$$x_{k+1}^n = x_k^n + a_k^n \Delta + \sum_{j=1}^{m} b_k^{n,j} \Delta W^j + \frac{1}{2} L^0 a_k^n \Delta^2 +$$

$$\sum_{j=1}^{m} \left[L^j a_k^n \Delta Z^j + \frac{\partial}{\partial t} b_k^{n,j} \left\{ \Delta W^j \Delta - \Delta Z^j \right\} \right]$$

$$\Delta W^j = \sqrt{\Delta}\xi_j,\ \Delta Z^j = \frac{\Delta}{2}\left(\sqrt{\Delta}\xi_j + \tilde{a}_{j,0} \right), \tilde{a}_{j,0} =$$

$$-\frac{\sqrt{2\Delta}}{\pi} \sum_{i=1}^{p} \frac{1}{i} \zeta_{j,i} - 2\sqrt{\Delta \rho_p} \mu_{j,p},$$

$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{i=1}^{p} \frac{1}{i^2}, L^0 = \frac{\partial}{\partial t_k} + \sum_{i=1}^{d} a_k^i \frac{\partial}{\partial x_k^i} +$$

$$\frac{1}{2} \sum_{i,l=1}^{d} \sum_{j=1}^{m} b_k^{i,j} b_k^{l,j} \frac{\partial^2}{\partial x_k^i \partial x_k^l}, L^j = \sum_{i=1}^{d} b_k^{i,j} \frac{\partial}{\partial x_k^i}$$

$$x(t)=\left\{u_1(t) \quad \dot{u}_1(t) \quad u_2(t) \quad \dot{u}_2(t) \quad z(t) \quad p_1(t) \quad p_2(t)\right\}^t$$

$$x_{k+1}^1 = x_k^1 + a_k^1\Delta + \frac{1}{2}L^0 a_k^1\Delta^2 + L^1 a_k^1 \Delta Z^1 + L^2 a_k^1 \Delta Z^2 + L^3 a_k^1 \Delta Z^3$$
$$+L^4 a_k^1 \Delta Z^4 + L^5 a_k^1 \Delta Z^5$$

$$x_{k+1}^2 = x_k^2 + a_k^2\Delta + \frac{\sigma_1}{m_1}\Delta W^1 + \frac{1}{2}L^0 a_k^2\Delta^2 + L^1 a_k^2 \Delta Z^1 + L^2 a_k^2 \Delta Z^2$$
$$+L^3 a_k^2 \Delta Z^3 + L^4 a_k^2 \Delta Z^4 + L^5 a_k^2 \Delta Z^5$$

$$x_{k+1}^3 = x_k^3 + a_k^3\Delta + \frac{1}{2}L^0 a_k^3\Delta^2 + L^1 a_k^3 \Delta Z^1 + L^2 a_k^3 \Delta Z^2 + L^3 a_k^3 \Delta Z^3$$
$$+L^4 a_k^3 \Delta Z^4 + L^5 a_k^3 \Delta Z^5$$

$$x_{k+1}^4 = x_k^4 + a_k^4\Delta + \frac{\sigma_2}{m_2}\Delta W^2 + \frac{1}{2}L^0 a_k^4 \Delta^2 + L^1 a_k^4 \Delta Z^1 + L^2 a_k^4 \Delta Z^2$$

$$+L^3 a_k^4 \Delta Z^3 + L^4 a_k^4 \Delta Z^4 + L^5 a_k^4 \Delta Z^5$$

$$x_{k+1}^5 = x_k^5 + a_k^5\Delta + \sigma_3 \Delta W^3 + \frac{1}{2}L^0 a_k^5 \Delta^2 + L^1 a_k^5 \Delta Z^1 + L^2 a_k^5 \Delta Z^2$$

$$+L^3 a_k^5 \Delta Z^3 + L^4 a_k^5 \Delta Z^4 + L^5 a_k^5 \Delta Z^5$$

$$x_{k+1}^6 = x_k^6 + a_k^6\Delta + \sigma_4 \Delta W^4 + \frac{1}{2}L^0 a_k^6 \Delta^2 + L^1 a_k^6 \Delta Z^1 + L^2 a_k^6 \Delta Z^2$$

$$+L^3 a_k^6 \Delta Z^3 + L^4 a_k^6 \Delta Z^4 + L^5 a_k^6 \Delta Z^5$$

$$x_{k+1}^7 = x_k^7 + a_k^7\Delta + \sigma_5 \Delta W^5 + \frac{1}{2}L^0 a_k^7 \Delta^2 + L^1 a_k^7 \Delta Z^1 + L^2 a_k^7 \Delta Z^2$$

$$+L^3 a_k^7 \Delta Z^3 + L^4 a_k^7 \Delta Z^4 + L^5 a_k^7 \Delta Z^5$$

$$L^0 a_k^1 = a_k^2, L^1 a_k^1 = \frac{\sigma_1}{m_1}, L^2 a_k^1 = 0, L^3 a_k^1 = 0, L^4 a_k^1 = 0, L^5 a_k^1 = 0$$

$$L^0 a_k^2 = -\frac{a_k^1}{m_1}\left[k_1 + k_2 + 3\alpha_1\left(x_k^1\right)^2 + 3\alpha_2\left(x_k^1 - x_k^3\right)^2\right] -$$

$$\frac{a_k^2}{m_1}\left[c_1 + c_2\right] - \frac{a_k^3}{m_1}\left[-k_2 - 3\alpha_2\left(x_k^1 - x_k^3\right)^2\right] - \frac{a_k^4}{m_1}\left[-c_2\right] + \frac{a_k^5}{m_1}$$

$$L^1 a_k^2 = -\left(\frac{\sigma_1}{m_1^2}\right)\left[c_1 + c_2\right], L^2 a_k^2 = -\left(\frac{\sigma_2}{m_2 m_1}\right)\left[-c_2\right], L^3 a_k^2 = 0,$$

$$L^4 a_k^2 = \frac{\sigma_4}{m_1}, L^5 a_k^2 = 0$$

$$L^0 a_k^3 = a_k^4, L^1 a_k^3 = 0, L^2 a_k^3 = \frac{\sigma_2}{m_2}, L^3 a_k^3 = 0, L^4 a_k^3 = 0, L^5 a_k^3 = 0$$

$$L^0 a_k^4 = -\frac{a_k^1}{m_2}\left[-k_2 - 3\alpha_2\left(x_k^3 - x_k^1\right)^2\right] - \frac{a_k^2}{m_2}\left[-c_2\right] -$$

$$\frac{a_k^3}{m_2}\left[k_2 + k_3\bar{\lambda} + 3\alpha_2\left(x_k^3 - x_k^1\right)^2\right] - \frac{a_k^4}{m_1}\left[c_2 + c_3\right] - \frac{a_k^5}{m_2}\left[k_3\left(1-\bar{\lambda}\right)\right]$$

$$L^1 a_k^4 = -\left(\frac{\sigma_1}{m_2 m_1}\right)\left[-c_2\right], L^2 a_k^4 = -\left(\frac{\sigma_2}{m_2^2}\right)\left[c_2 + c_3\right],$$

$$L^3 a_k^4 = \frac{\bar{\sigma}_3}{m_2} k_3\left(1-\bar{\lambda}\right), L^4 a_k^4 = 0, L^5 a_k^4 = \frac{\sigma_5}{m_2}$$

$$L^0 a_k^5 = a_k^4\left[-\gamma x_k^5 |x_k^5|^{\bar{n}-1}\operatorname{sgn}\left(x_k^4\right)-\beta |x_k^5|^{\bar{n}} + A\right]$$

$$+a_k^5\left[-\gamma |x_k^4|| x_k^5|^{\bar{n}-1} - \gamma\left(\bar{n}-1\right)|x_k^4|x_k^5|x_k^5|^{\bar{n}-2}\operatorname{sgn}\left(x_k^5\right)-\beta\bar{n}x_k^4|x_k^5|^{\bar{n}-1}\operatorname{sgn}\left(x_k^5\right)\right]$$

$$+\frac{\sigma_2^2}{2m_2^2}\left[-2\gamma\delta\left(x_k^4\right)x_k^5|x_k^5|^{\bar{n}-1}\right]$$

$$+\frac{\sigma_3^2}{2}\left[\begin{array}{l}-\gamma\left(\bar{n}-1\right)|x_k^4||x_k^5|^{\bar{n}-2}\operatorname{sgn}\left(x_k^5\right)-\gamma\left(\bar{n}-1\right)\left(\bar{n}-2\right)|x_k^4|x_k^5|x_k^5|^{\bar{n}-3}\left(\operatorname{sgn}\left(x_k^5\right)\right)^2 \\ -\gamma\left(\bar{n}-1\right)|x_k^4||x_k^5|^{\bar{n}-2}\operatorname{sgn}\left(x_k^5\right)-2\gamma\left(\bar{n}-1\right)|x_k^4|x_k^5|x_k^5|^{\bar{n}-2}\delta\left(x_k^5\right) \\ -\beta\bar{n}\left(\bar{n}-1\right)x_k^4|x_k^5|^{\bar{n}-2}\left(\operatorname{sgn}\left(x_k^5\right)\right)^2-2\beta\bar{n}x_k^4|x_k^5|^{\bar{n}-1}\delta\left(x_k^5\right)\end{array}\right]$$

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

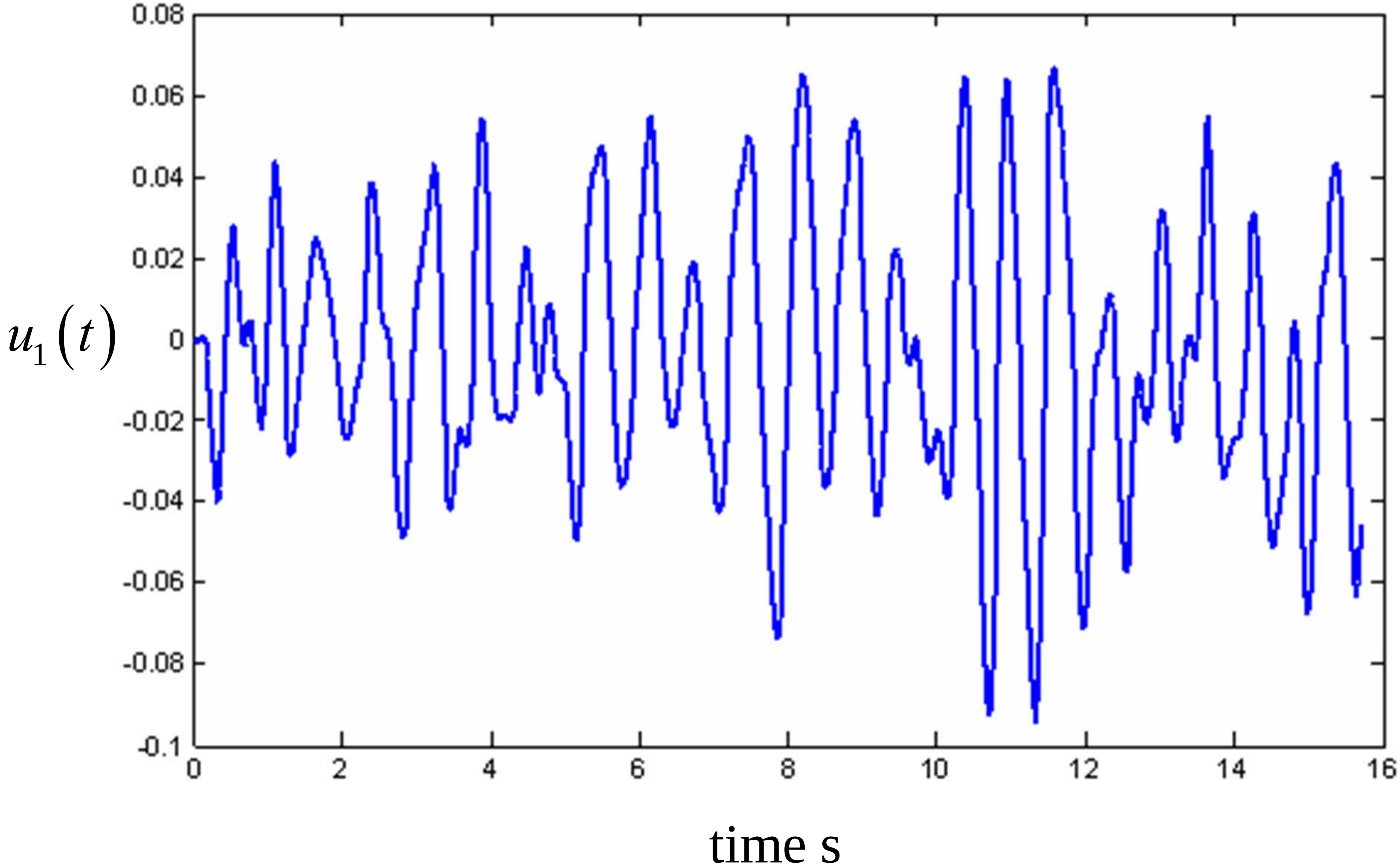
$$L^1 a_k^5 = 0, L^2 a_k^5 = \left(\frac{\sigma_2}{m_2^2}\right)\left[-\gamma x_k^5 \,|\, x_k^5 \,|^{\bar{n}-1} \operatorname{sgn}\left(x_k^4\right) - \beta \,|\, x_k^5 \,|^{\bar{n}} + A\right],$$

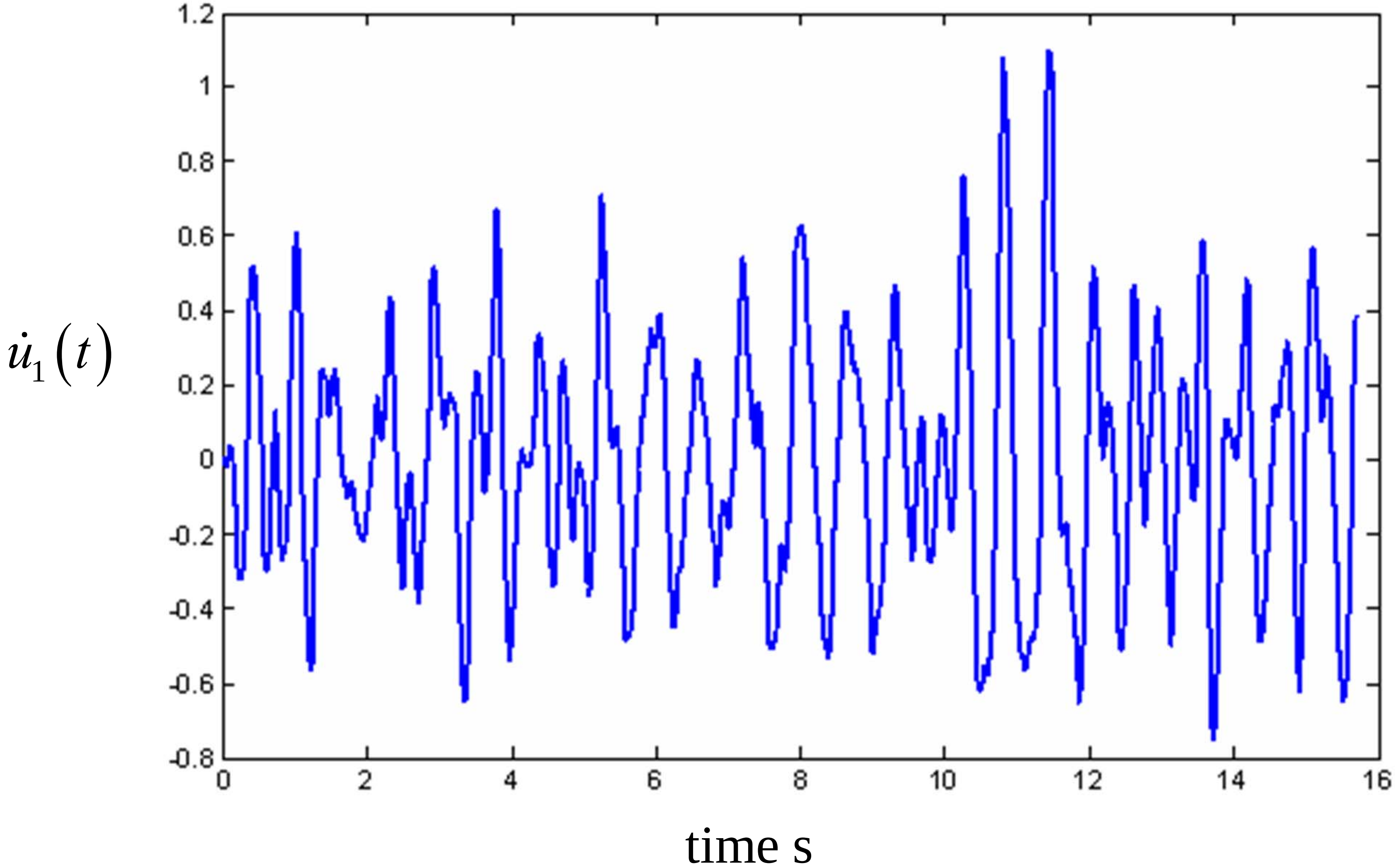
$$L^3 a_k^5 = \sigma_3\left[-\gamma \,|\, x_k^4 \,||\, x_k^5 \,|^{\bar{n}-1} - \gamma\left(\bar{n}-1\right) |\, x_k^4 \,|\, x_k^5 \,|\, x^5 \,|^{\bar{n}-2} \operatorname{sgn}\left(x_k^5\right) - \beta \bar{n} x_k^4 \,|\, x_k^5 \,|^{\bar{n}-2} \operatorname{sgn}\left(x_k^5\right)\right],$$

$$L^4 a_k^5 = 0, L^5 a_k^5 = 0$$

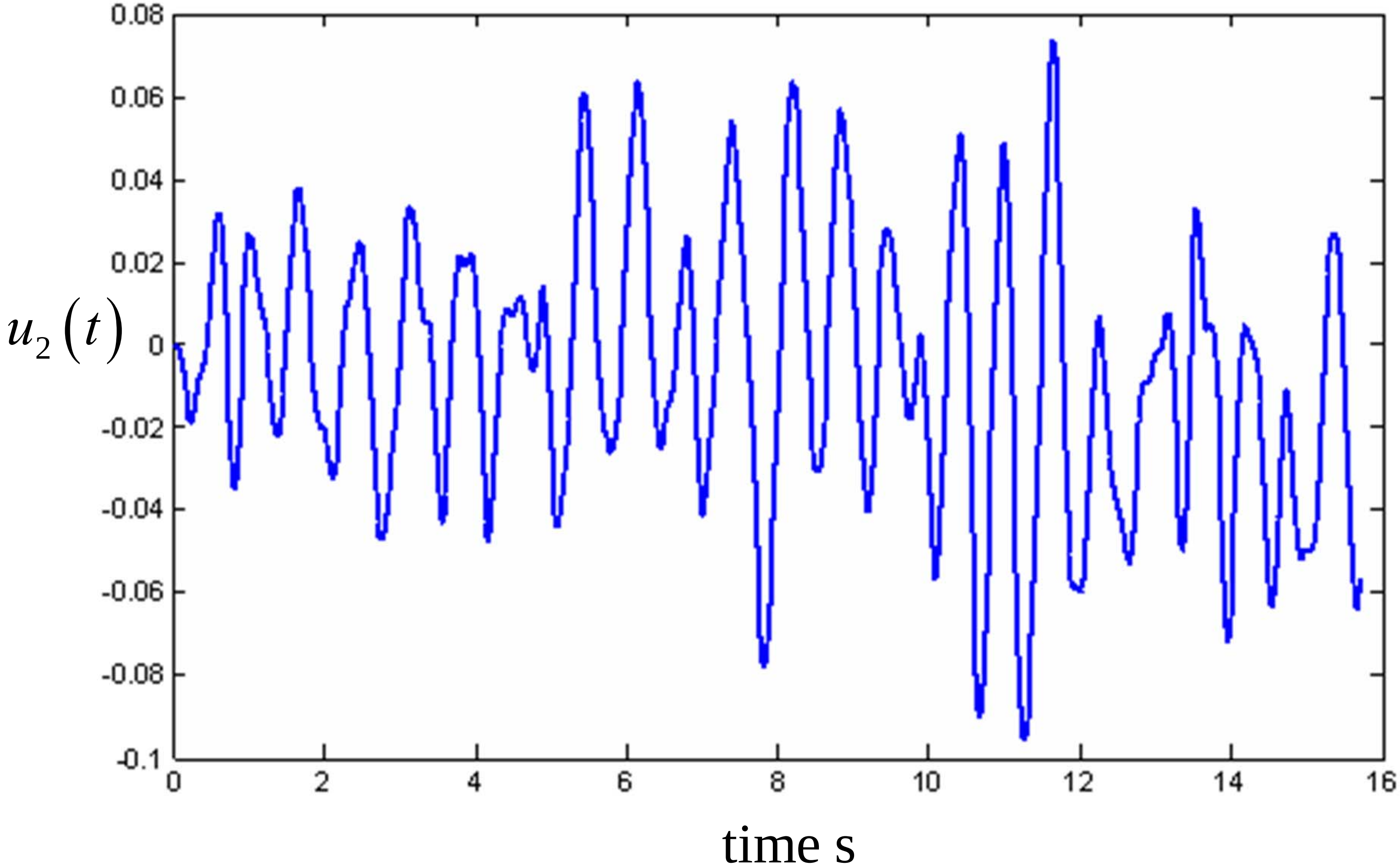
$$L^0 a_k^6 = -a_k^6 \bar{\alpha}_1, L^1 a_k^6 = 0, L^2 a_k^6 = 0, L^3 a_k^6 = 0, L^4 a_k^6 = -\sigma_4 \bar{\alpha}_1, L^5 a_k^6 = 0$$

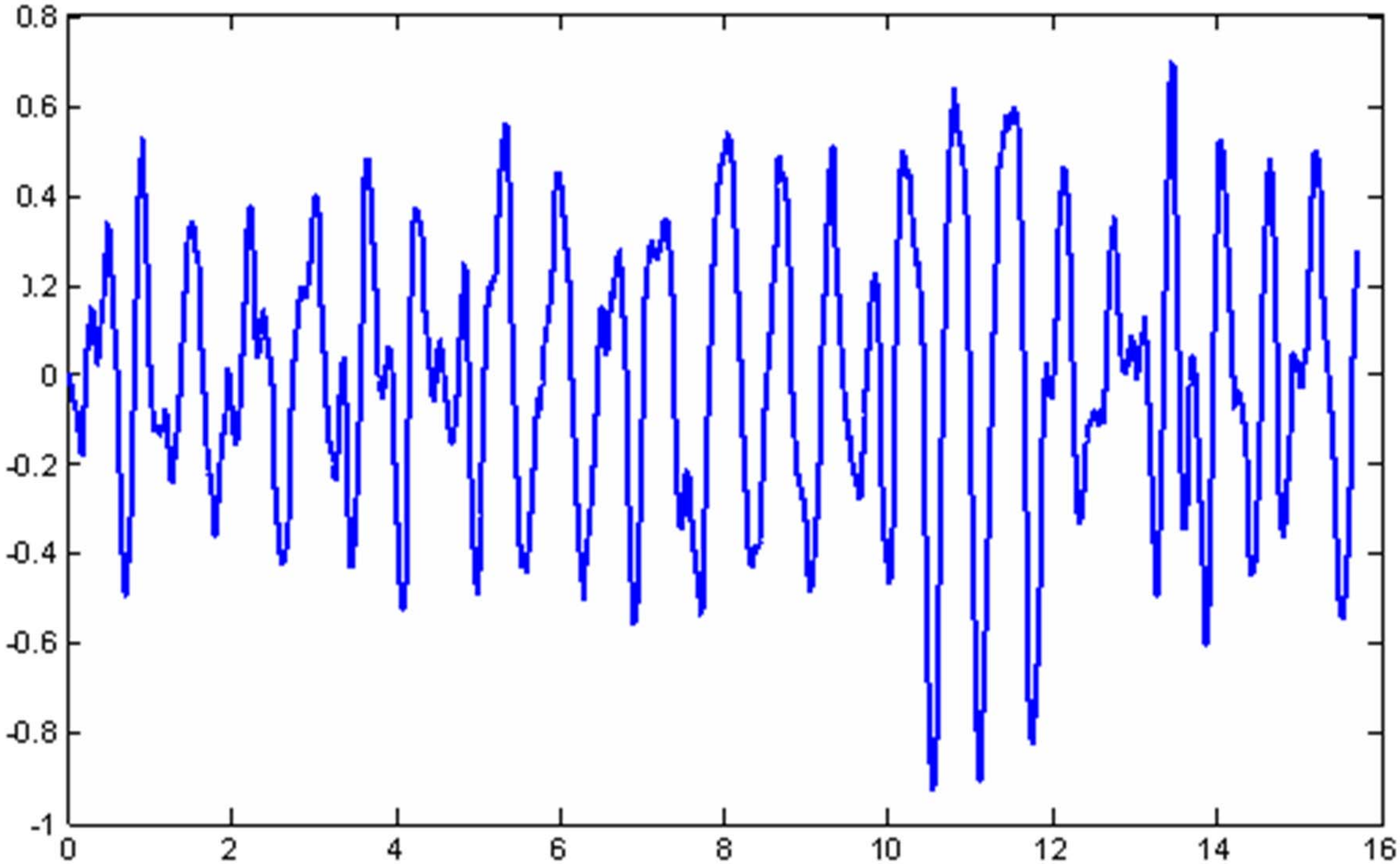
$$L^0 a_k^7 = -a_k^7 \bar{\alpha}_2, L^1 a_k^7 = 0, L^2 a_k^7 = 0, L^3 a_k^7 = 0, L^4 a_k^7 = 0, L^5 a_k^7 = -\sigma_4 \bar{\alpha}_2$$


0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
2
4
6
8
10
12
14
16
$u_1(t)$
time s



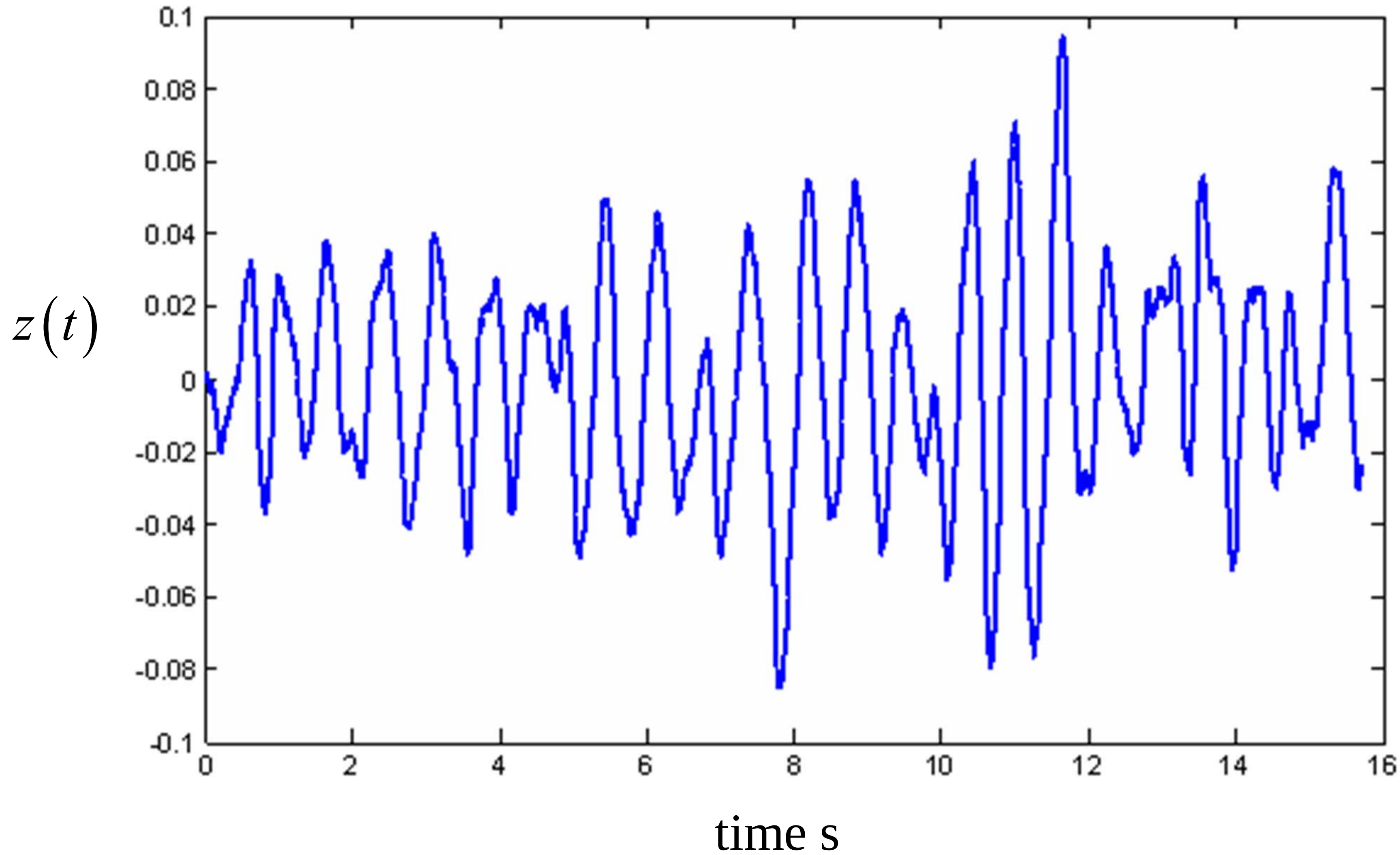
$\dot{u}_1(t)$
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
2
4
6
8
10
12
14
16
time s

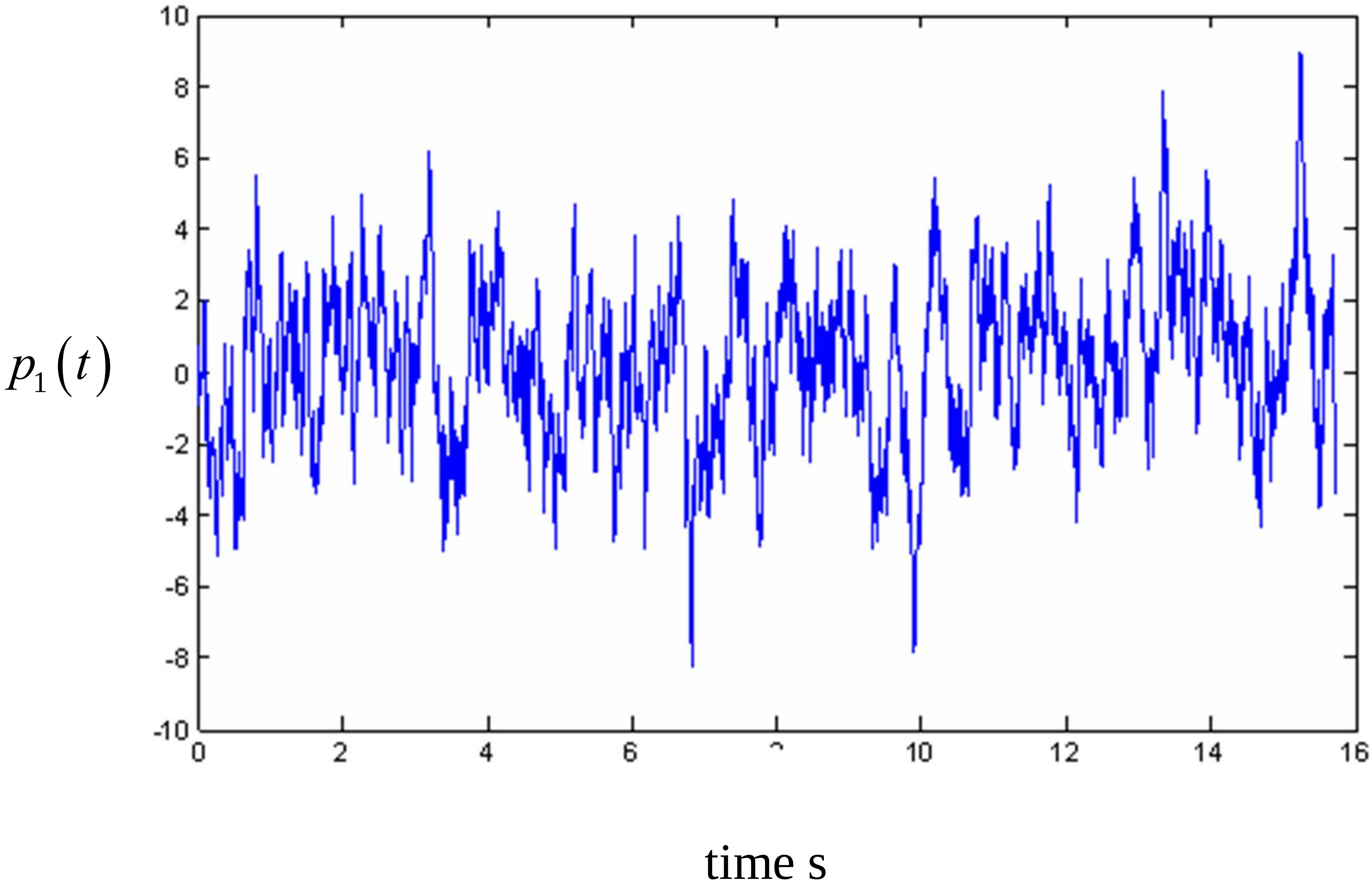

$u_2(t)$
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
2
4
6
8
10
12
14
16
time s



$\dot{u}_2(t)$

time s


0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
2
4
6
8
10
12
14
16
$z(t)$
time s


$p_1(t)$
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
2
4
6
10
12
14
16
time s

Stochastic structural dynamics

Uncertainty modeling
- Probability
- Random variables
- Random processes

Propagation of uncertainty
- Analytical procedures for LTI systems (time/frequency)
- Markov vector approach

Reliability analysis
- First passage
- Extremes
- Fatigue

Monte Carlo simulations
- Gaussian/non-Gaussian Random variables & random processes
- Variance reduction

Applications
- Earthquake
- Fatigue
- High frequency analysis

What next?

•Structural system identification
•Reliability model updating
•Structural health monitoring

Further applications
•Wind, waves, guideway uneveneness,…
•Hazard and risk analysis
  •Earthquake
  •Wind
•Performance based design
•Design code development

…..

Sensing + Computing

Discussion on mean square estimation

Introductory comments

Let X and Y be two random variables with a known jpdf. Assuming that in a particular experiment, the random variable Y can be measured and takes the value y. What can we say about the corresponding value, say x, of the unobservable variable X?

Suppose we make an estimate, say, x^*, of the value of X when $Y=y$, according to the rule $x^*=h(y)$. $h(y)$=unspecified function of y. The error of our estimate $e=x-h(y)$.
We can never hope to make $e=0$.
Can we select h such that we minimize the expected value of some function of e?

Typical problem in dealing with random processes

Let *x(t)* and *y(t)* be two Gaussian random processes with a known joint pdf.
Let it be assumed that we can observe *y(t)* and not *x(t)*.
Given the observation of a sample of *y(t)* for *t* in *0* to *T*, how to estimate the value of *x(t) for some value of t*?

Problem 1

Let *Y* be a random variable and *c* be a constant. We wish to estimate *Y* by a constant.
Find *c* such that E[(*Y-c)*^{2}] is minimized.

$$e = E\left[(Y-c)^2\right] = \int_{-\infty}^{\infty} (y-c)^2 p_Y(y)dy$$

$$\frac{\partial e}{\partial c} = 0 \Rightarrow c = \int_{-\infty}^{\infty} y p_Y(y)dy = E[Y]$$

Problem 2

Let X and Y be two random variables. We wish to estimate Y by a function $c(X)$.

To find $c(X)$ such that e=E{[Y-$c(X)$]^2} is minimized.

$$e = E\left\{\left[Y - c(X)\right]^2\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y - c(x)\right]^2 p_{XY}(x, y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[y - c(x)\right]^2 p_Y(y \mid X = x)p_X(x)dxdy$$

$$= \int_{-\infty}^{\infty}\underbrace{p_X(x)}_{\geq 0}\left[\int_{-\infty}^{\infty}\underbrace{\left[y - c(x)\right]^2}_{\geq 0} p_Y(y \mid X = x)dy\right]dx$$

e would be a minimum if $c(x)$ minimizes

$$\int_{-\infty}^{\infty}\left[y - c(x)\right]^2 p_Y(y \mid X = x)dy \text{ for every fixed value of } x.$$

From solution of Problem 1 we have

$$c(x) = \int_{-\infty}^{\infty} y p_Y(y \mid X = x) dy = E\left[Y \mid X = x\right]$$

Remarks

- If $Y=g(X)$, $c(x)=E[g(X)|X=x]=g(x)$ & $e=0$.
- If X and Y are independent, $c(x)=\langle Y\rangle$=constant

Linear MS estimation

Let $c(X) = AX + B$

$$\Rightarrow e = E\left[\left(Y - AX - B\right)^2\right]$$

$$\frac{\partial e}{\partial B} = 0 \Rightarrow E\left[\left(Y - AX - B\right)\right] = 0 \Rightarrow B = \eta_Y - A\eta_X$$

$$\Rightarrow e = E\left[\left(Y - AX - \eta_Y + A\eta_X\right)^2\right] = E\left[\left\{\left[Y - \eta_Y\right] - A\left[X - \eta_X\right]\right\}^2\right]$$

$$\frac{\partial e}{\partial A} = 0 \Rightarrow E\left[\left(Y - \eta_Y\right)\left(X - \eta_X\right) - A\left(X - \eta_X\right)^2\right] = 0$$

$$\Rightarrow A = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{r_{XY}\sigma_X\sigma_Y}{\sigma_X^2} = \frac{r_{XY}\sigma_Y}{\sigma_X}$$

$$e_{\min} = E\left[\left\{Y - r_{XY}\frac{\sigma_Y}{\sigma_X}X - \eta_Y + \frac{\eta_X r_{XY}\sigma_Y}{\sigma_X}\right\}^2\right]$$

$$= E\left[\left\{\left(Y - \eta_Y\right) - r_{XY}\frac{\sigma_Y}{\sigma_X}\left(X - \eta_X\right)\right\}^2\right] = \sigma_Y^2\left(1 - r_{XY}^2\right)$$

$$A = r_{XY}\frac{\sigma_Y}{\sigma_X} \qquad B = \eta_Y - A\eta_X = \eta_Y - \eta_X r_{XY}\frac{\sigma_Y}{\sigma_X}$$

$$e_{\min} = \sigma_Y^2\left(1 - r_{XY}^2\right)$$

Let X and Y be Gaussian

$$\Rightarrow c(x) = E\left[Y \mid X = x\right] = \frac{r_{XY}\sigma_Y}{\sigma_X}x - \frac{r_{XY}\sigma_Y\eta_X}{\sigma_X}$$

For normal random variables, linear and nonlinear ms estimation lead to identical results.

The orthogonality principle

$$e = E\left[\left\{Y - \left(AX + B\right)\right\}^2\right]$$

$$\frac{\partial e}{\partial A} = 0 \Rightarrow E\left[\left\{Y - \left(AX + B\right)\right\} X\right] = 0$$

$$Y - \left(AX + B\right) = \text{Error}$$

$$X = \text{data}$$

Data is orthogonal to error

General case of linear ms estimation

Let S be a random variable &

$$\hat{S} = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n = a^t X$$

be an estimatior of S.

$$P = E\left[\left(S - \hat{S}\right)^2\right] = E\left[\left\{S - \left(a_1 X_1 + a_2 X_2 + \cdots + a_n X_n\right)\right\}^2\right]$$

Select $\left\{a_i\right\}_{i=1}^{n} \ni \dfrac{\partial P}{\partial a_i} = 0 \forall i = 1, 2, \cdots, n$

$$\langle SX_1 \rangle = a_1 \langle X_1^2 \rangle + a_2 \langle X_1 X_2 \rangle + \cdots + a_{n1} \langle X_1 X_n \rangle$$

$$\langle SX_2 \rangle = a_1 \langle X_1 X_2 \rangle + a_2 \langle X_2^2 \rangle + \cdots + a_{n1} \langle X_2 X_n \rangle$$

$$\vdots$$

$$\langle SX_n \rangle = a_1 \langle X_1 X_n \rangle + a_2 \langle X_2 X_n \rangle + \cdots + a_{n1} \langle X_n^2 \rangle$$

$$\Rightarrow$$

$$\begin{Bmatrix} R_{01} \\ R_{02} \\ R_{03} \\ \vdots \\ R_{0n} \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & \cdots & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & \cdots & R_{2n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ R_{n1} & R_{n2} & & & R_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{Bmatrix}$$

$$R_0 = RA \Rightarrow A = R^{-1} R_0$$ //

We have $E\left[\left\{S-\hat{S}\right\}X_i\right]=0\,\forall i\in[1,n]$

$$\Rightarrow E\left[\left\{S-\hat{S}\right\}\{a_1X_1+a_2X_2+\cdots+a_nX_n\}\right]=0$$

$$\Rightarrow E\left[\left\{S-\hat{S}\right\}\hat{S}\right]=0\Rightarrow\left(S-\hat{S}\right)\perp\hat{S}$$

$$P=E\left[\left(S-\hat{S}\right)\left(S-\hat{S}\right)\right]=E\left[\left(S-\hat{S}\right)S\right]-E\left[\left(S-\hat{S}\right)\hat{S}\right]$$

$$\Rightarrow P=E\left[\left(S-\hat{S}\right)S\right]=E\left[S^2\right]-E\left[S\hat{S}\right]$$

$$E\left[S\hat{S}\right]=E\left[S\left(a_1X_1+a_2X_2+\cdots+a_nX_n\right)\right]$$

$$=a_1R_{01}+a_2R_{02}+\cdots+a_nR_{0n}$$

$$=A^tR_0$$

$$\Rightarrow P=E\left[S^2\right]-A^tR_0$$

Nonlinear estimation

$$P = E\left[\left\{S - g(X_1, X_2, \cdots, X_n)\right\}^2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{s - g(x_1, x_2, \cdots, x_n)\right\}^2 p_{S\tilde{X}}(s, \tilde{x})\, ds\, d\tilde{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{s - g(x_1, x_2, \cdots, x_n)\right\}^2 p_S(s \mid \tilde{X} = \tilde{x})\, p_{\tilde{X}}(\tilde{x})\, ds\, d\tilde{x}$$

$$= \int_{-\infty}^{\infty} \underbrace{p_{\tilde{X}}(\tilde{x})}_{\geq 0} \left[\int_{-\infty}^{\infty} \underbrace{\left\{s - g(x_1, x_2, \cdots, x_n)\right\}^2}_{\geq 0} p_S(s \mid \tilde{X} = \tilde{x})\, ds\right] d\tilde{x}$$

P is minimum when the second integrand is minimum for any $\tilde{x}$

$$\Rightarrow g(x_1, x_2, \cdots, x_n) = E\left[S \mid \tilde{X} = \tilde{x}\right]$$

General orthogonality principle

We have

$$E\left[\left\{S-(a_1X_1+a_2X_2+\cdots+a_nX_n)\right\}X_i\right]=0\,\forall i\in[1,n]$$

i.e., $E\left[\left\{S-\hat{S}\right\}X_i\right]=0\,\forall i\in[1,n]$

$$E\left[\left\{S-\hat{S}\right\}\left(c_1X_1+c_2X_2+\cdots+c_nX_n\right)\right]=0 \text{ for any } c_i\,\forall i\in[1,n]$$

If $g(X)$ is the nonlinear ms estimator of S, the estimation error $S-g(X)$ is orthogonal to any function $w(X)$, linear or nonlinear function of data.

Proof :

$$E\left[\{S-g(X)\}w(X)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[s-g(x)]w(x)p_{SX}(s,x)dsdx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[s-g(x)]w(x)p_S(s\,|\,x)p_X(x)dsdx$$

$$=\int_{-\infty}^{\infty}w(x)\left\{\int_{-\infty}^{\infty}[s-g(x)]p_S(s\,|\,x)ds\right\}p_X(x)dx$$

$$=\int_{-\infty}^{\infty}w(x)E\left[\{S-g(X)\}\,|\,X=x\right]p_X(x)dx$$

$$=\int_{-\infty}^{\infty}w(x)\left\{E\left(S\,|\,X=x\right)-g(x)\right\}p_X(x)dx$$

$$=\int_{-\infty}^{\infty}w(x)\{g(x)-g(x)\}p_X(x)dx=0\ QED$$

If S and X are jointly normal it can be shown that linear and nonlinear estimations of S are equal.

Exercise

Estimation of a random process

Let $S(t)$ & $X(\xi)$ be two random processes with $a \le \xi \le b$.

Consider the problem of estimating $S(t)$ for a fixed t in terms of $X(\xi)$ specified for every ξ in an interval $a \le \xi \le b$ of finite or infinite length. $X(\xi)$ = data available.

$$\hat{S}(t) = \int_a^b X(\alpha)h(\alpha)d\alpha$$

$$P = E\left[\left\{S(t) - \int_a^b X(\alpha)h(\alpha)d\alpha\right\}^2\right]$$

Select $h(\alpha)$ such that P is minimized.

$$\hat{S}(t) \cong \sum_{k=1}^{n} h(\alpha_k)X\left(\alpha_k\right)\Delta\alpha$$

$$\text{error} = S(t) - \sum_{k=1}^{m} h(\alpha_k)X\left(\alpha_k\right)\Delta\alpha$$

Orthogonality principle $\Rightarrow$

$$E\left[\left\{S(t) - \sum_{k=1}^{m} h(\alpha_k)X\left(\alpha_k\right)\Delta\alpha\right\}X(\xi_j)\right] = 0 \forall j \in [1, m]$$

$$\Rightarrow R_{SX}(t, \xi_j) = \sum_{k=1}^{m} h(\alpha_k)R_{XX}(\alpha_k, \xi_j)\Delta\alpha \forall j \in [1, m]$$

$$\lim_{\Delta\alpha\to 0} R_{SX}(t,\xi) = \int_a^b h(\alpha) R_{XX}(\alpha,\xi)\, d\alpha$$

Smoothing: $a \le t \le b$

Prediction: $X(t) = S(t); t \notin [a,b]$

Forward prediction: $t > b$

Backward prediction: $t < a$

Filtering: $X(t) \ne S(t)$

Problem: Let $S(t)$ be a stationary random process.

Estimate $S(t+\alpha)$ in terms of $S(t)$.

$$\hat{S}(t+\lambda) = aS(t)$$

$$P = E\left[\left\{S(t+\lambda) - aS(t)\right\}^2\right]$$

$$\frac{\partial P}{\partial a} = 0 \Rightarrow E\left[\left\{S(t+\lambda) - aS(t)\right\} S(t)\right] = 0$$

$$\Rightarrow a = \frac{R_{SS}(t+\lambda, t)}{R_{SS}(t,t)} = \frac{R_{SS}(\lambda)}{\sigma_S^2}$$

$$P = E\left[\left\{S(t+\lambda) - aS(t)\right\}\left\{S(t+\lambda) - aS(t)\right\}\right]$$

$$= E\left[\left\{S(t+\lambda) - aS(t)\right\}S(t+\lambda)\right]$$

$$= R_{SS}(0) - aR_{SS}(\lambda)$$

$$= R_{SS}(0) - \frac{aR_{SS}^2(\lambda)}{R_{SS}(0)}$$

$$\text{Let } R_{SS}(\tau) = A\exp\left[-\alpha\,|\,\tau\,|\right]$$

$$\Rightarrow a = \frac{A\exp\left[-\alpha\,|\,\lambda\,|\right]}{A} = \exp\left[-\alpha\,|\,\lambda\,|\right]$$

Example : Let $\hat{S}(t+\lambda) = a_1 S(t) + a_2 \dot{S}(t)$

Note: $\left[S(t+\lambda) - a_1 S(t) - a_2 \dot{S}(t) \right] \perp S(t), \dot{S}(t)$

$$E\left[S(t)\dot{S}(t) \right] = 0$$

$$P = E\left[\left\{ S(t+\lambda) - a_1 S(t) - a_2 \dot{S}(t) \right\}^2 \right]$$

$$\frac{\partial P}{\partial a_1} = 0 \Rightarrow R_{SS}(t+\lambda, t) - a_1 R_{SS}(t,t) = 0$$

$$\Rightarrow a_1 = \frac{R_{SS}(t+\lambda, t)}{R_{SS}(t,t)} = \frac{R_{SS}(\lambda)}{\sigma_S^2}$$

$$\frac{\partial P}{\partial a_2} = 0 \Rightarrow R_{S\dot{S}}(t+\lambda, t) - a_2 R_{\dot{S}\dot{S}}(t,t) = 0 \Rightarrow a_2 = \frac{R_{S\dot{S}}(t+\lambda, t)}{R_{\dot{S}\dot{S}}(t,t)}$$

$$
\begin{aligned}
P &= E\left[\left\{S(t+\lambda) - a_1 S(t) - a_2 \dot{S}(t)\right\}\left\{S(t+\lambda) - a_1 S(t) - a_2 \dot{S}(t)\right\}\right] \\
&= E\left[\left\{S(t+\lambda) - a_1 S(t) - a_2 \dot{S}(t)\right\} S(t+\lambda)\right] \\
&= R_{SS}(0) - a_1 R_{SS}(\lambda) - a_2 R_{S\dot{S}}(t+\lambda, t) \quad //
\end{aligned}
$$

Filtering

$$\hat{S}(t) = aX(t)$$

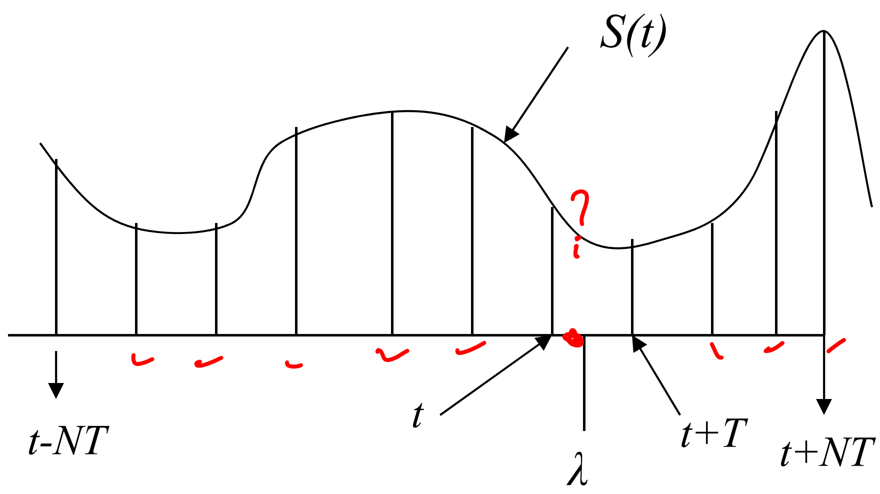
$$P = E\left[\left\{S(t) - aX(t)\right\}^2\right]$$

$$\frac{\partial P}{\partial a} = 0 \Rightarrow a = \frac{R_{SX}(0)}{R_{XX}(0)}$$

$$P = E\left[\left\{S(t) - aX(t)\right\} S(t)\right]$$

$$= R_{SS}(0) - \frac{R_{SX}^2(0)}{R_{XX}(0)}$$

Interpolation



To estimate *s(t+ λ)* in the interval t to *t+T* in terms of samples of *S(t), S(t+kT), k=-N,-(N-1),…,0,1,2,…,N*

$$\hat{S}(t+\lambda) = \sum_{k=-N}^{N} a_k S(t+kT); \quad 0 < \lambda < T$$

$$P = E\left[\left\{S(t+\lambda) - \sum_{k=-N}^{N} a_k S(t+kT)\right\}^2\right]$$

$$\frac{\partial P}{\partial a_j} = 0 \Rightarrow E\left[\left\{S(t+\lambda) - \sum_{k=-N}^{N} a_k S(t+kT)\right\} S(t+jT)\right] \forall j \in [-N, N]$$

Set of $2N+1$ equations for $\{a_k\}_{k=-N}^{N}$

Quadrature

$$Z=\int_0^b S(t)dt$$

$$\hat{Z}=a_0S(0)+a_1S(T)+\cdots+a_NS(NT);\quad T=\frac{b}{N}$$

$$P=E\left[\left\{\int_0^b S(t)dt-\left(a_0S(0)+a_1S(T)+\cdots+a_NS(NT)\right)\right\}^2\right]$$

$$\frac{\partial P}{\partial a_k}=0\Rightarrow\int_0^b E\left[S(t)S(kT)\right]dt-a_0E\left[S(0)S(kT)\right]-$$

$$a_1E\left[S(T)S(kT)\right]\cdots-a_nE\left[S(nT)S(kT)\right]=0\forall j\in\left[0,N\right]$$

$N+1$ equations in $N+1$ unknowns

Smoothing

Estimate present value of $S(t)$ in terms of values of $X(\xi)$ for $-\infty < \xi < \infty$

$$X(t) = S(t) + \upsilon(t)$$

$$\hat{S}(t) = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha$$

$$S(t) - \hat{S}(t) \perp X(\xi) \forall \xi \in (-\infty, \infty)$$

$$E\left[\left\{S(t) - \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha\right\} X(t-\tau)\right] = 0$$

$$\Rightarrow R_{SX}(\tau) = \int_{-\infty}^{\infty} h(\alpha) R_{XX}(\tau-\alpha) d\alpha$$

Probelm of dynamic state estimation

Process equation

$$x_{k+1} = \varphi_k x_k + w_k$$ // ← FE

Measurement equation

$$z_k = H_k x_k + \upsilon_k$$ // Sensors

$x_k : n \times 1$ state evector

$w_k : n \times 1$ process noise; iid sequence N(0,1)

$\varphi_k : n \times n$ state transistion matrix

$z_k : m \times 1$ measurement vector

$H_k : m \times n$ relates states to measurements

$\upsilon_k : m \times 1$ measurement noise; iid N(0,1)

Probelm of dynamic state estimation

Determine

$$p\left(x_{0:k} \mid z_{1:k}\right)$$

$x_{0:k} = \{x_0, x_1, x_2 \cdots x_k\}$

$$p\left(x_k \mid z_{1:k}\right)$$

— filter pdf

$$a_{k|k} = \left\langle x_k \mid z_{1:k} \right\rangle$$

$$\Sigma_{k|k} = \left\langle \left[x_k - a_{k|k}\right]\left[x_k - a_{k|k}\right] \mid z_{1:k} \right\rangle$$

Kalman filter provides the exact solution to this problem