Stochastic Structural Dynamics

Lecture-38

Problem solving session-2

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Problem 17

Consider the vector random variable Y given by

$Y=\{Y_1 \quad Y_2 \quad Y_3\}^t$. It is given that Y is normal with

mean vector μ and correlation matrix R given by

$$\mu=\begin{Bmatrix}1\\2\\3\end{Bmatrix} \text{ and } R=\langle YY^t\rangle=\begin{bmatrix}4 & -1 & 6\\-1 & 9 & 0\\6 & 0 & 19\end{bmatrix}.$$

We now form the random process

$$X(t)=Y_1+Y_2t+Y_3t^2.$$

Find the mean, autocorrelations and cross correlations of $X(t)$ and $\dot{X}(t)$.

$$\text{Define } N(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix}^t \Rightarrow X(t) = N^t(t) Y$$

$$\langle X(t) \rangle = N^t(t) \langle Y \rangle = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = 1 + 2t + 3t^2$$

$$\langle \dot{X}(t) \rangle = \langle B + 2Ct \rangle = 2 + 6t$$

$$\langle X(t_1) X^t(t_2) \rangle = N^t(t_1) \langle YY^t \rangle N(t_2)$$

$$= \begin{bmatrix} 1 & t_1 & t_1^2 \end{bmatrix}^t \begin{bmatrix} 4 & -1 & 6 \\ -1 & 9 & 0 \\ 6 & 0 & 19 \end{bmatrix} \begin{Bmatrix} 1 \\ t_2 \\ t_2^2 \end{Bmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 \end{bmatrix}^t \begin{Bmatrix} 4 - t_2 + 6t_2^2 \\ -1 + 9t_2 \\ 6 + 19t_2^2 \end{Bmatrix}$$

$$R_{XX}(t_1, t_2) = 4 - t_1 - t_2 + 9t_1 t_2 + 6t_1^2 + 6t_2^2 + 19t_1^2 t_2^2$$

$$\langle X^2(t) \rangle = 4 - 2t + 21t^2 + 19t^4$$

$$\langle X(t)\rangle = 1 + 2t + 3t^2$$

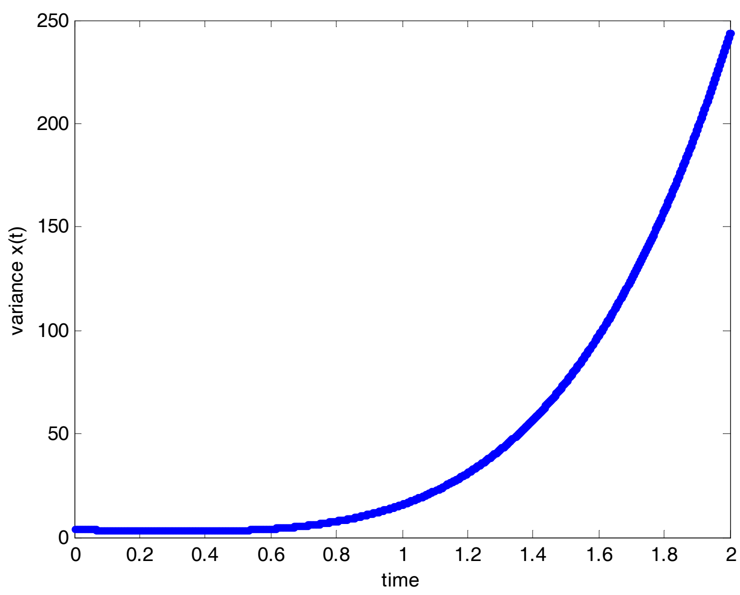
$$R_{XX}(t_1, t_2) = 4 - t_1 - t_2 + 9t_1t_2 + 6t_1^2 + 6t_2^2 + 19t_1^2t_2^2$$

$$\langle X^2(t)\rangle = 4 - 2t + 21t^2 + 19t^4$$

$$\sigma_X^2(t) = 4 - 2t + 21t^2 + 19t^4 - \left(1 + 2t + 3t^2\right)^2$$

$$= 4 - 2t + 21t^2 + 19t^4 - (1 + 4t^2 + 9t^4 + 4t + 6t^2 + 12t^3)$$

$$= 4 - 6t + 11t^2 - 12t^3 + 19t^4$$



$$R_{XX}\left(t_1,t_2\right)=4-t_1-t_2+9t_1t_2+6t_1^2+6t_2^2+19t_1^2t_2^2$$

$$\left\langle X\left(t_1\right)\dot{X}\left(t_2\right)\right\rangle=\frac{\partial}{\partial t_2}R_{XX}\left(t_1,t_2\right)=-1+9t_1+12t_2+38t_1^2t_2$$

Check

$$\left\langle X\left(t_1\right)\dot{X}\left(t_2\right)\right\rangle=\left\langle\left(Y_1+Y_2t_1+Y_3t_1^2\right)\left(Y_2+2Y_3t_2\right)\right\rangle$$

$$=-1+12t_2+9t_1+38t_1^2t_2\quad(ok)$$

$$\left\langle \dot{X}\left(t_1\right)\dot{X}\left(t_2\right)\right\rangle=\frac{\partial^2}{\partial t_1\partial t_2}R_{XX}\left(t_1,t_2\right)=9+76t_1t_2$$

Problem 18

Consider the random process $X(t) = a\exp\left[j\left(\Omega t - \Theta\right)\right]$

where $a =$ deterministic constant, $j = \sqrt{-1}$,

$\Omega =$ is a random variable with pdf $p_{\Omega}(\omega)$ and

characteristic function $\Phi_{\Omega}(\lambda)$, and

Θ=a random variable that is independent of Ω and

distributed uniformly in $(-\pi, \pi)$. Show that

- $R_{XX}(\tau)$ is proportional to $\Phi_{\Omega}(\lambda)$, and
- $S_{XX}(\omega)$ is proportional to $p_{\Omega}(\omega)$

$$X(t) = a\exp\left[j(\Omega t - \Theta)\right]$$

$$\left\langle a\exp(j\Omega t - \Theta)\right\rangle = a\left\langle \exp(j\Omega t)\right\rangle\left\langle \exp(-j\Theta)\right\rangle = 0$$

$$\left\langle X(t)X^*(t-\lambda)\right\rangle$$

$$= a^2\left\langle \exp\left[j(\Omega t - \Theta)\right]\exp\left[-j(\Omega t - \Omega\lambda - \Theta)\right]\right\rangle$$

$$= a^2\left\langle \exp\left[j\Omega\lambda\right]\right\rangle$$

$$\Rightarrow R_{XX}(\lambda) = a^2\Phi_\Omega(\lambda)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} a^2\Phi_\Omega(\lambda)\exp(-j\omega\lambda)\,d\lambda = a^2 p_\Omega(\omega)$$

Problem 19: A random process $Y(t)$ is given by

$$Y(t) = X(t) + 2X(t-\tau) + X(t-2\tau)$$

where $X(t)$ is a zero mean stationary random process with PSD function

$$S_{XX}(\omega) = \frac{C}{\omega^2 + \alpha^2}$$

Determine the PSD function of $Y(t)$.

$$Y(t) = X(t) + 2X(t - \lambda) + X(t - 2\lambda)$$

$$\Rightarrow \left\langle Y(t) \right\rangle = \left\langle X(t) + 2X(t - \tau) + X(t - 2\tau) \right\rangle = 0$$

$$Y(t+\tau) = X(t+\tau) + 2X(t - \lambda + \tau) + X(t - 2\lambda + \tau)$$

$$\left\langle Y(t)Y(t+\tau) \right\rangle = R_{XX}(\tau) + 2R_{XX}(-\lambda+\tau) + R_{XX}(-2\lambda+\tau)$$

$$+2R_{XX}(\tau+\lambda) + 4R_{XX}(\tau) + 2R_{XX}(-\lambda+\tau)$$

$$+R_{XX}(\tau+2\lambda) + 2R_{XX}(\lambda+\tau) + R_{XX}(\tau)$$

$$R_{YY}(\tau) = 6R_{XX}(\tau) + 4R_{XX}(\tau-\lambda) + 4R_{XX}(\tau+\lambda)$$

$$+R_{XX}(\tau-2\lambda) + R_{XX}(\tau+2\lambda)$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\exp(-i\omega\tau)\,d\tau$$

Consider $\int_{-\infty}^{\infty} R(\tau + a)\exp(-i\omega\tau)\,d\tau$

$$= \int_{-\infty}^{\infty} R(u)\exp(-i\omega(u-a))\,d\tau$$

$$= \exp(i\omega a)\int_{-\infty}^{\infty} R(u)\exp(-i\omega u)\,d\tau = \exp(i\omega a)\,S_{UU}(\omega)$$

$$\int_{-\infty}^{\infty} R(\tau)\exp(-i\omega\tau)\,d\tau = S(\omega)$$

$$\int_{-\infty}^{\infty} R(\tau + a)\exp(-i\omega\tau)\,d\tau = \exp(i\omega a)\,S_{UU}(\omega)$$

$$R_{YY}(\tau) = 6R_{XX}(\tau) + 4R_{XX}(\tau - \lambda) + 4R_{XX}(\tau + \lambda)$$

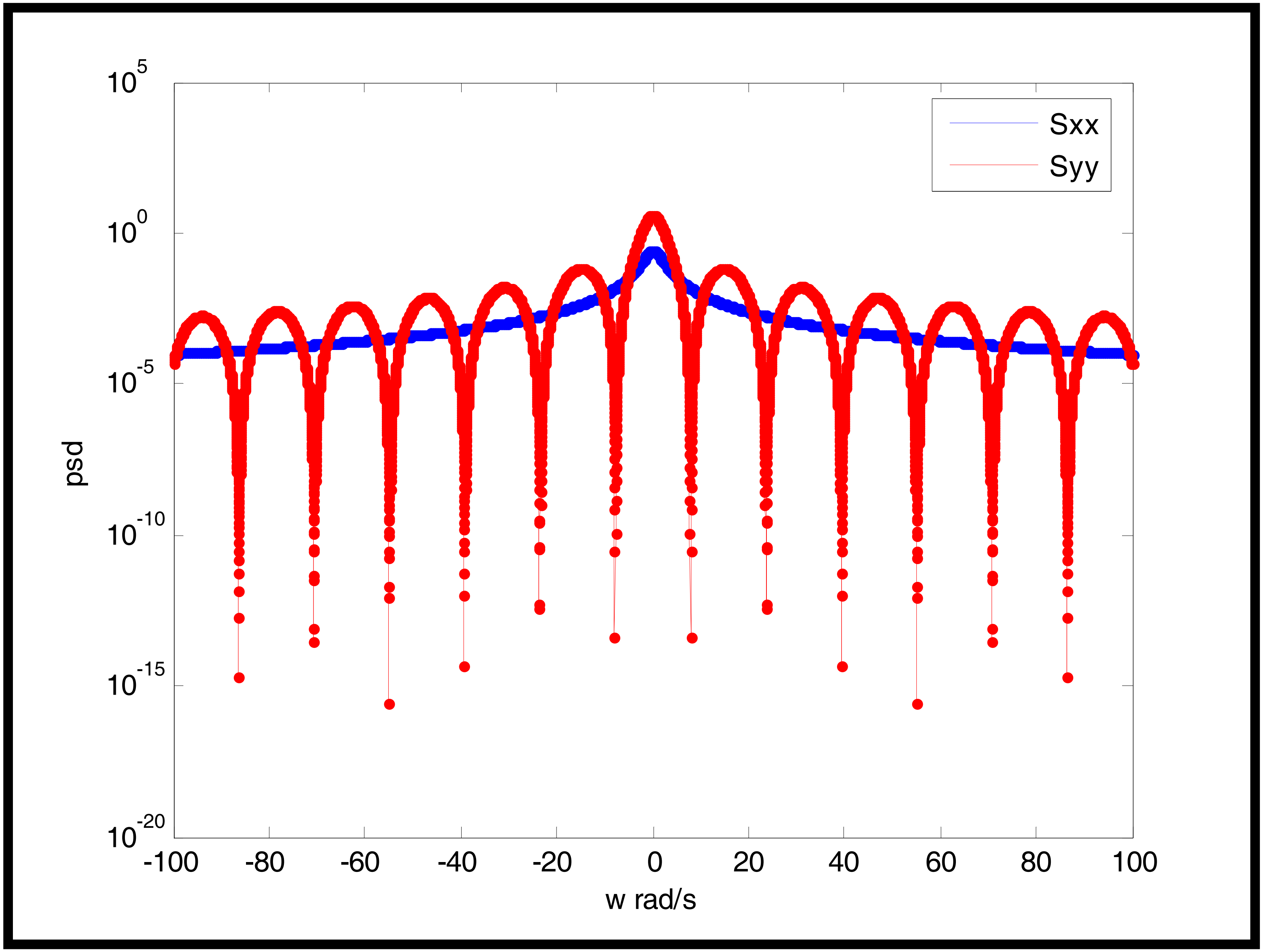
$$+R_{XX}(\tau - 2\lambda) + R_{XX}(\tau + 2\lambda)$$

$$\Rightarrow S_{YY}(\omega) = 6S_{XX}(\omega) + 4S_{XX}(\omega)\left[\exp(i\omega\lambda) + \exp(-i\omega\lambda)\right]$$

$$+S_{XX}(\omega)\left[\exp(2i\omega\lambda) + \exp(-2i\omega\lambda)\right]$$

$$= S_{XX}(\omega)\left[6 + 8\cos(\omega\lambda) + 2\cos(2\omega\lambda)\right]$$

$$S_{YY}(\omega) = \frac{C}{\omega^2 + \alpha^2}\left[6 + 8\cos(\omega\lambda) + 2\cos(2\omega\lambda)\right]$$



10^5
10^0
10^-5
10^-10
10^-15
10^-20
psd
-100
-80
-60
-40
-20
0
20
40
60
80
100
w rad/s
Sxx
Syy

Problem 20

Consider two independent random processes $X(t)$ and $Y(t)$ which have zero mean and are stationary. Define $Z(t) = X(t)Y(t-\lambda)$ where λ is a deterministic constant. Determine PSD function of $Z(t)$.

$$Z(t) = X(t)Y(t-\lambda)$$

$$\Rightarrow \langle Z(t)\rangle = \langle X(t)Y(t-\lambda)\rangle = \langle X(t)\rangle\langle Y(t-\lambda)\rangle = 0$$

$$\langle Z(t)Z(t+\tau)\rangle = \langle X(t)Y(t-\lambda)X(t+\tau)Y(t-\lambda+\tau)\rangle$$

$$= \langle X(t)X(t+\tau)Y(t-\lambda+\tau)\rangle\langle Y(t-\lambda)Y(t-\lambda+\tau)\rangle$$

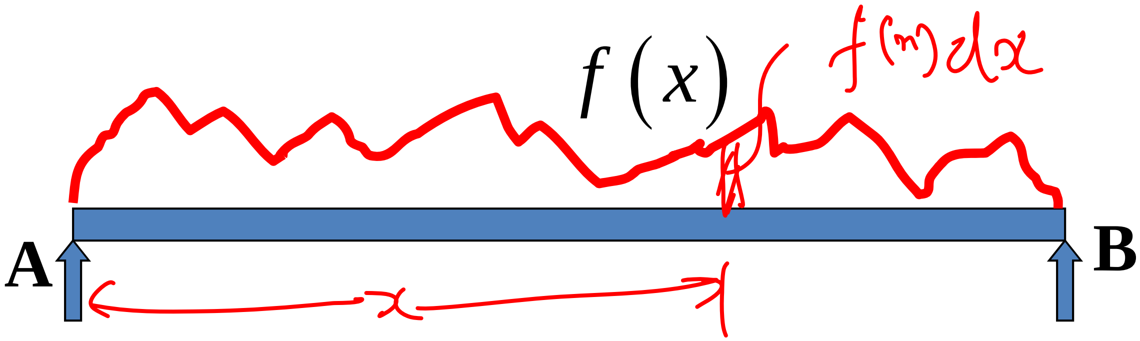
$$R_{ZZ}(\tau) = R_{XX}(\tau)R_{YY}(\tau)$$

$$\Rightarrow S_{ZZ}(\omega) = \int_{-\infty}^{\infty} S_{XX}(\varsigma)S_{YY}(\varsigma-\omega)\,d\varsigma$$

Problem 21

A simply supported beam of span L carries a distributed load $f(x)$. The load is modeled as a segment of stationary random process as $f(x) = F_0\left[1+\varepsilon\xi(x)\right]$ such that $\langle\xi(x)\rangle = 0$ and $\langle\xi(x)\xi(x+\tau)\rangle = \delta(\tau)$. Determine the following

- Bending moment at midspan
- Joint pdf of reactions at the two supports.



$$R_B L = \int_0^L x f(x)\, dx$$

$$R_B = \frac{1}{L}\int_0^L x f(x)\, dx = \frac{1}{L}\int_0^L x F_0 \left[1 + \varepsilon \xi(x)\right] dx$$

$$= \frac{F_0 L}{2} + \frac{F_0 \varepsilon}{L}\int_0^L x \xi(x)\, dx \quad //$$

$$\Rightarrow \langle R_B \rangle = \left\langle \frac{F_0 L}{2} \right\rangle + \frac{F_0 \varepsilon}{L}\int_0^L x \langle \xi(x) \rangle\, dx = \frac{F_0 L}{2}$$

$$\left(R_B - \frac{F_0 L}{2}\right) = \frac{F_0 \varepsilon}{L}\int_0^L x\xi(x)\,dx$$

$$\left\langle \left(R_B - \frac{F_0 L}{2}\right)^2 \right\rangle = \left(\frac{F_0 \varepsilon}{L}\right)^2 \int_0^L\int_0^L x_1 x_2 \left\langle \xi(x_1)\xi(x_2)\right\rangle dx_1 dx_2$$

$$= \left(\frac{F_0 \varepsilon}{L}\right)^2 \int_0^L\int_0^L x_1 x_2 I_0 \delta(x_1 - x_2)\,dx_1 dx_2$$

$$= \left(\frac{F_0 \varepsilon}{L}\right)^2 \int_0^L x^2 I_0\,dx = \frac{F_0^2 \varepsilon^2 L I_0}{3}$$

$$R_B \sim N\left[\frac{F_0 L}{2} \quad \sqrt{\frac{F_0^2 \varepsilon^2 L I_0}{3}}\right]$$

$$R_A L = \int_0^L (L-x)\, f(x)\, dx$$

$$R_A = \frac{1}{L}\int_0^L (L-x)\, f(x)\, dx = \frac{1}{L}\int_0^L (L-x)\, F_0\left[1+\varepsilon\xi(x)\right] dx$$

$$= \frac{F_0 L}{2} + \frac{F_0 \varepsilon}{L}\int_0^L (L-x)\,\xi(x)\, dx$$

$$\langle R_A \rangle = \frac{F_0 L}{2} \;\&\; \left\langle \left(R_A - \frac{F_0 L}{2} \right)^2 \right\rangle = \frac{F_0^{\,2}\varepsilon^2 L I_0}{3}$$

$$\Rightarrow R_B \sim N\left[\frac{F_0 L}{2} \quad \sqrt{\frac{F_0^{\,2}\varepsilon^2 L I_0}{3}}\right]$$

$$\left\langle \left(R_A - \frac{F_0 L}{2} \right) \left(R_B - \frac{F_0 L}{2} \right) \right\rangle$$

$$= \frac{F_0^{\,2} \varepsilon^2}{L^2} \int_0^L \int_0^L (L - x_1) x_2 \langle \xi(x_1) \xi(x_2) \rangle dx_1 dx_2$$

$$= \frac{F_0^{\,2} \varepsilon^2}{L^2} \int_0^L \int_0^L (L - x_1) x_2 I_0 \delta(x_2 - x_1) dx_1 dx_2$$

$$= \frac{F_0^{\,2} \varepsilon^2}{L^2} I_0 \int_0^L x(L - x) dx = \frac{F_0^{\,2} \varepsilon^2 I_0 L}{6}$$

$$\begin{pmatrix} R_A \\ R_B \end{pmatrix} \sim N \left[\begin{pmatrix} \frac{F_0 L}{2} \\ \frac{F_0 L}{2} \end{pmatrix} \begin{pmatrix} \frac{F_0^{\,2} \varepsilon^2 L I_0}{3} & \frac{F_0^{\,2} \varepsilon^2 L I_0}{6} \\ \frac{F_0^{\,2} \varepsilon^2 L I_0}{6} & \frac{F_0^{\,2} \varepsilon^2 L I_0}{3} \end{pmatrix} \right]$$

Similarly one can study

$$M\left(\frac{L}{2}\right)=M_0=\frac{R_A L}{2}-\int_0^{\frac{L}{2}}\left(\frac{L}{2}-x\right)f(x)dx$$

$$M_0=\frac{L}{2}\frac{1}{L}\int_0^{L}\left(L-x\right)f\left(x\right)dx-\int_0^{\frac{L}{2}}\left(\frac{L}{2}-x\right)f(x)dx$$

$$=\frac{1}{2}\int_0^{L}\left(L-x\right)F_0\left[1+\varepsilon\xi\left(x\right)\right]dx-\int_0^{\frac{L}{2}}\left(\frac{L}{2}-x\right)f(x)dx$$

Exercise: complete the characterization of M_0

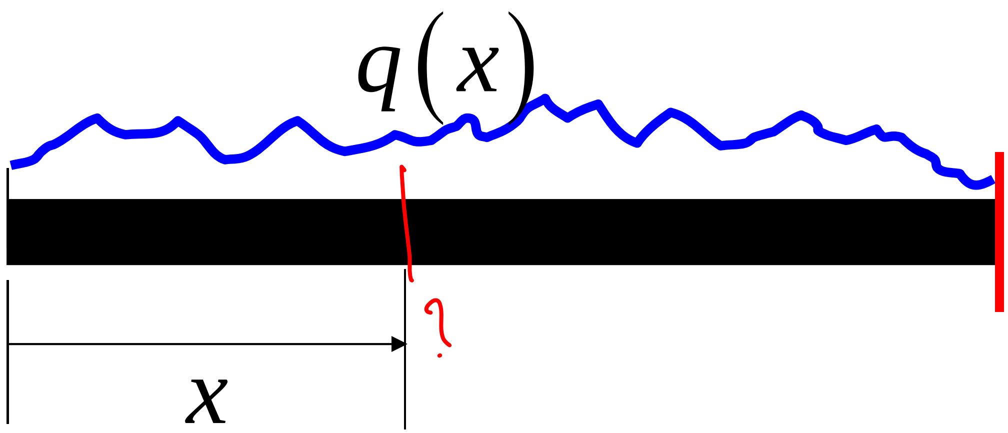
Problem 22

A cantilever beam carries a randomly distributed load as shown below. The load $q(x)$ is modeled as

$$q(x) = q_0\left[1+\varepsilon f(x)\right]; \langle f(x)\rangle = 0 \;\&$$

$$\langle f(x_1) f(x_2)\rangle = \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right]$$

Determine the bending moment at a section x measured from the free end.



$$M(x) = \int_0^x (x-\xi) q(\xi)\, d\xi$$

$$= \int_0^x (x-\xi) q_0 \left[1 + \varepsilon f(\xi)\right] d\xi$$

$$\langle M(x) \rangle = \int_0^x (x-\xi) q_0\, d\xi = q_0 \left[x^2 - \frac{x^2}{2}\right] = q_0 \frac{x^2}{2}$$

$$\left\langle \left[M(x) - q_0 \frac{x^2}{2} \right]^2 \right\rangle =$$

$$\left\langle q_0^2 \varepsilon^2 \int_0^x \int_0^x (x-\xi_1)(x-\xi_2) \langle f(\xi_1) f(\xi_2) \rangle d\xi_1 d\xi_2 \right\rangle$$

$$= q_0^2 \varepsilon^2 \int_0^x \int_0^x (x-\xi_1)(x-\xi_2) \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)\right] d\xi_1 d\xi_2$$

$$\sigma_M^2(x) = q_0^2\varepsilon^2 \int_0^x\int_0^x (x-\xi_1)(x-\xi_2)\frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\xi_1^2+\xi_2^2\right)\right]d\xi_1 d\xi_2$$

$$= q_0^2\varepsilon^2 x^2 \int_0^x\int_0^x \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\xi_1^2+\xi_2^2\right)\right]d\xi_1 d\xi_2$$

$$-q_0^2\varepsilon^2 x \int_0^x\int_0^x \xi_2 \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\xi_1^2+\xi_2^2\right)\right]d\xi_1 d\xi_2$$

$$-q_0^2\varepsilon^2 x \int_0^x\int_0^x \xi_1 \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\xi_1^2+\xi_2^2\right)\right]d\xi_1 d\xi_2$$

$$+q_0^2\varepsilon^2 \int_0^x\int_0^x \xi_1\xi_2 \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\xi_1^2+\xi_2^2\right)\right]d\xi_1 d\xi_2$$

$$\sigma_M^2(x) = q_0^2\varepsilon^2 x^2\Psi^2(x) - 2xq_0^2\varepsilon^2 \frac{\Psi(x)}{\sqrt{2\pi}}\left[1-\exp\left(-\frac{x^2}{2}\right)\right]$$

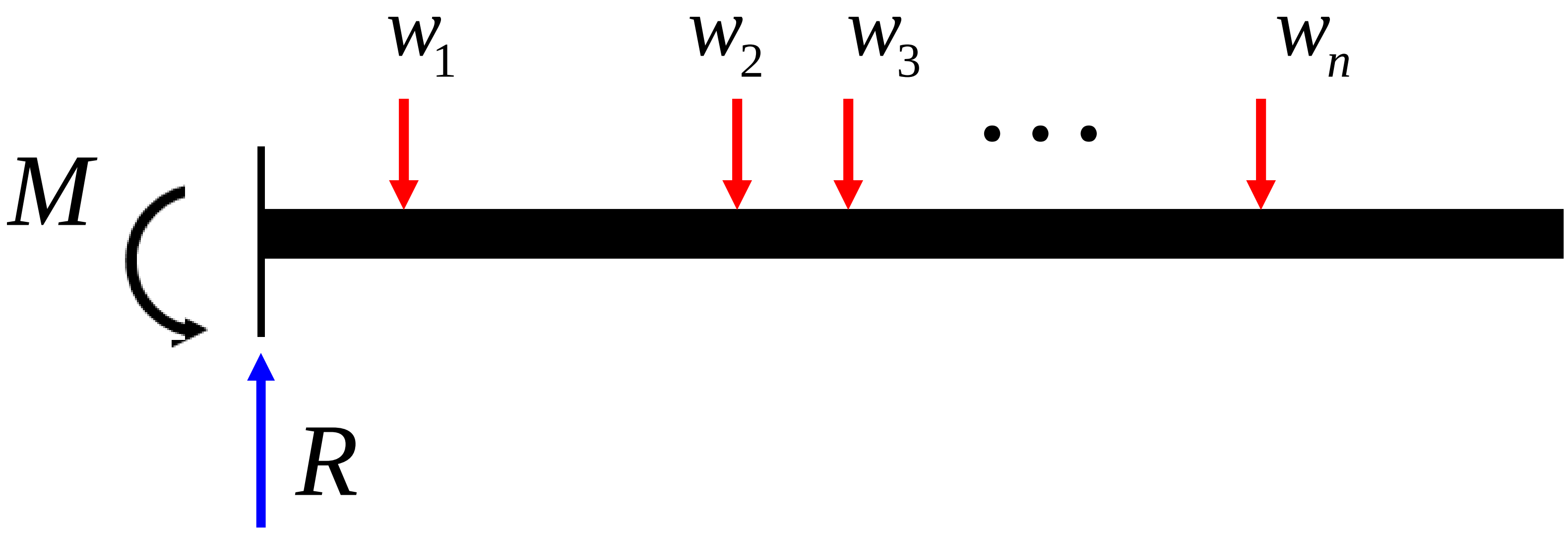
$$+\frac{q_0^2\varepsilon^2}{2\pi}\left[1-\exp\left(-\frac{x^2}{2}\right)\right]^2$$

with

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp\left(-\frac{x^2}{2}\right)dx$$

Problem 23

A cantilever beam of span L carries a series of concentrated loads. The point of application of these loads are distributed as Poisson points on 0 to L. The magnitude of the loads are modeled as a sequence of iid-s with a common Rayleigh distribution with paramter σ. Determine the characteristic funciton of the reaction R.



$$R = \sum_{n=1}^{N(L)} w_n$$

$$P\left[N(L) = n\right] = \exp(-aL)\frac{(aL)^n}{n!}; n = 0, 1, 2, \cdots$$

$$p_w(w) = \frac{w}{\sigma^2}\exp\left(-\frac{w^2}{2\sigma^2}\right); w \geq 0$$

$$R = \sum_{n=1}^{N(L)} w_n \Rightarrow \Phi_R(\omega) = \langle \exp[i\omega R] \rangle = \left\langle \exp\left[i\omega \sum_{n=1}^{N(L)} w_i \right] \right\rangle$$

$$= P[N(L)=0] + \sum_{k=1}^{\infty} \left\{ \exp\left[i\omega \sum_{n=1}^{k} w_i \right] \middle\| N(L)=k \right\} P[N(L)=k]$$

$$= \exp(-aL) + \sum_{k=1}^{\infty} \{\Phi_w(i\omega)\}^k \exp(-aL) \frac{(aL)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \{\Phi_w(i\omega)\}^k \frac{(aL)^k}{k!} \exp(-aL) = \exp\left[aL \{\Phi_w(i\omega) - 1\} \right]$$

Note: $\Phi_w(i\omega) = 1 + i\omega \exp\left(-\frac{\sigma^2 \omega^2}{2} \right) \left[1 + \mathrm{erf}\left(\frac{i\sigma\omega}{\sqrt{2}} \right) \right]$ (prove)

Problem 24

Verify if

$$R(t_1,t_2)=\beta\exp\left[-\alpha|t_2-t_1|\right]\sin\left(\gamma|t_2-t_1|\right)$$

can be a valid autocovariance function of a zero mean random process. It is given that $\alpha,\beta,\gamma\geq 0$.

Required characteristics

- $R(t_1,t_2)=R(t_2,t_1)$
- $R(t,t)>0$
- $|R(t_1,t_2)|\leq R(t_1,t_1)R(t_2,t_2)$

No, since the function is not positive definite; notice $R(t,t)=0$

Problem 25

Consider $X(t)$ to be a stationary random process with zero mean. Define $Y(t) = X(t) + a(t)\dot{X}(t-\lambda)$ where λ and $a(t)$ are determinisitc. Determine autocovariance of $Y(t)$. It is given that

$R_{XX}(\tau) = \sigma^2 \exp(-\alpha|\tau|)[1+\beta|\tau|]$

$$Y(t) = X(t) + a(t)\dot{X}(t - \lambda)$$

$$\langle Y(t) \rangle = \left\langle X(t) + a(t)\dot{X}(t - \lambda) \right\rangle = 0$$

$$\left\langle Y(t)Y(t+\tau) \right\rangle =$$

$$\left\langle \left[X(t) + a(t)\dot{X}(t - \lambda) \right] \left[X(t+\tau) + a(t+\tau)\dot{X}(t+\tau-\lambda) \right] \right\rangle$$

$$= \left\langle X(t)X(t+\tau) \right\rangle + \left\langle X(t)a(t+\tau)\dot{X}(t+\tau-\lambda) \right\rangle$$

$$+ \left\langle a(t)\dot{X}(t-\lambda)X(t+\tau) \right\rangle$$

$$+ \left\langle a(t)\dot{X}(t-\lambda)a(t+\tau)\dot{X}(t+\tau-\lambda) \right\rangle$$

$$= R_{XX}(\tau) + a(t+\tau)R_{X\dot{X}}(\tau-\lambda) + a(t)R_{\dot{X}X}(\tau+\lambda)$$

$$+ a(t)a(t+\tau)R_{\dot{X}\dot{X}}(\tau)$$

Simplify using $\left\langle \frac{d^n X(t+\tau)}{dt^n} \frac{d^m Y(t)}{dt^m} \right\rangle = (-1)^m \frac{d^{n+m} R_{XY}(\tau)}{d\tau^{n+m}}$

$$R_{XX}(\tau) = \sigma^2 \exp\left(-\alpha|\tau|\right)\left[1+\beta|\tau|\right]$$

$$\frac{d}{d\tau} R_{XX}(\tau) = -\alpha\sigma^2 \exp\left(-\alpha|\tau|\right)\operatorname{sgn}(\tau)\left[1+\beta|\tau|\right]$$

$$+\sigma^2 \exp\left(-\alpha|\tau|\right)\beta \operatorname{sgn}(\tau)$$

$$= \sigma^2 \exp\left(-\alpha|\tau|\right)\left[-\alpha \operatorname{sgn}(\tau) + \operatorname{sgn}(\tau)\beta|\tau| + \beta \operatorname{sgn}(\tau)\right]$$

$$= \sigma^2 \exp\left(-\alpha|\tau|\right)\operatorname{sgn}(\tau)\left[\beta - \alpha + \beta|\tau|\right]$$

Use

$$\operatorname{sgn}(\tau) = U(\tau) - U(-\tau) \;\&\; \frac{dU(t)}{dt} = \delta(t)$$

and derive $\dfrac{d^2}{d\tau^2} R_{XX}(\tau)$

Problem 26

An undamped sdof system is set into free vibration by imparting random initial displacement and velocity. Characterize the system response. Determine the conditions under which the response can become stationary.

$$\ddot{x}+\omega^2 x=0; x(0)=u; \dot{x}(0)=v$$

$$\Rightarrow x(t)=A\cos\omega t+B\sin\omega t$$

$$\Rightarrow x(t)=u\cos\omega t+\frac{v}{\omega}\sin\omega t$$

$$\langle x(t)\rangle=\langle u\rangle\cos\omega t+\frac{\langle v\rangle}{\omega}\sin\omega t$$

$$\langle x(t)x(t+\tau)\rangle=$$

$$\left\langle\left[u\cos\omega t+\frac{v}{\omega}\sin\omega t\right]\left[u\cos\omega(t+\tau)+\frac{v}{\omega}\sin\omega(t+\tau)\right]\right\rangle$$

$$R_{xx}(t,\tau)=\langle u^2\rangle\cos\omega t\cos\omega(t+\tau)+\frac{\langle uv\rangle}{\omega}\cos\omega t\sin\omega(t+\tau)$$

$$+\frac{\langle uv\rangle}{\omega}\sin\omega t\cos\omega(t+\tau)+\frac{\langle v^2\rangle}{\omega^2}\sin\omega t\sin\omega(t+\tau)$$

$$R_{xx}(t,\tau)=\langle u^2\rangle\cos\omega t\cos\omega(t+\tau)+\frac{\langle uv\rangle}{\omega}\cos\omega t\sin\omega(t+\tau)$$

$$+\frac{\langle uv\rangle}{\omega}\sin\omega t\cos\omega(t+\tau)+\frac{\langle v^2\rangle}{\omega^2}\sin\omega t\sin\omega(t+\tau)$$

Take $\langle uv\rangle=0\;\&\;\langle u^2\rangle=\frac{\langle v^2\rangle}{\omega^2}=\sigma^2$

$$\Rightarrow R_{xx}(t,\tau)=\sigma^2\left[\cos\omega t\cos\omega(t+\tau)+\sin\omega t\sin\omega(t+\tau)\right]$$

$$R_{xx}(t,\tau)=R_{xx}(\tau)=\sigma^2\cos\omega\tau$$

$\Rightarrow$ Conditions for existence of stochastic steady state are

$$\langle u\rangle=0,\langle v\rangle=0,\langle uv\rangle=0\;\&\;\langle u^2\rangle=\frac{\langle v^2\rangle}{\omega^2}=\sigma^2$$

Problem 27

An input $f(t) = 0$ for $t < 0$ and $f(t) = \exp(-2t)$ for $t \geq 0$ to a linear system produces the output $y(t) = \frac{1}{2}\left[\exp(-2t) - \exp(-4t)\right]$. The system is now excited by a Gaussian white noise excitation with unit strength. Determine the PSD of the steady state response.

$$f(t) = \exp(-2t)U(t)$$

$$\Rightarrow F(\omega) = \frac{1}{2+i\omega}$$

$$y(t) = \frac{1}{2}\left[\exp(-2t) - \exp(-4t)\right]U(t)$$

$$\Rightarrow Y(\omega) = \frac{1}{2}\left[\frac{1}{2+i\omega} - \frac{1}{4+i\omega}\right] =$$

$$H(\omega) = \frac{\frac{1}{2}\left[\frac{1}{2+i\omega} - \frac{1}{4+i\omega}\right]}{\frac{1}{2+i\omega}} = \frac{1}{2}\left[1 - \frac{2+i\omega}{4+i\omega}\right] = \frac{1}{4+i\omega}$$

$$S_{YY}(\omega) = \left|H(\omega)\right|^2 = \frac{1}{16+\omega^2} \quad //$$

Notice

$$\dot{x} + \beta x = \exp(-\alpha t); x(0) = 0$$

$$\Rightarrow x(t) = A\exp(-\beta t) + \frac{\exp(-\alpha t)}{\beta - \alpha}$$

$$x(0) = 0 \Rightarrow A = -\frac{1}{\beta - \alpha}$$

$$\Rightarrow x(t) = \frac{1}{\beta - \alpha}\left[\exp(-\alpha t) - \exp(-\beta t)\right]$$

$$\Rightarrow H(\omega) = \frac{1}{\beta + i\omega}$$

Problem 28

Consider a random process

$X(t) = P\sin(t+\Phi) + Y(t)$

where P is determinstic, Φ is a random variable distributed uniformly in 0 to 2π, and $Y(t)$ is a zero mean stationary Gaussian random process.

It may be assumed that $Y(t)$ and Φ are independent.

Determine the joint pdf of $X(t)$ and $\dot{X}(t)$.

Are $X(t)$ and $\dot{X}(t)$ uncorrelated? independent?

$$X(t) = F\sin(t+\Phi) + Y(t)$$

$$\langle X(t) \rangle = 0$$

$$\langle X(t)X(t+\tau) \rangle = \langle [F\sin(t+\Phi) + Y(t)][F\sin(t+\tau+\Phi) + Y(t+\tau)] \rangle$$

$$= F^2 \langle \sin(t+\Phi)\sin(t+\tau+\Phi) \rangle + R_{YY}(\tau)$$

$$= \frac{F^2}{2}\cos\tau + R_{YY}(\tau)$$

$$\Rightarrow X(t) \text{ is wide sense stationary}$$

$$X(t) = F\sin(t+\Phi) + Y(t)$$

$$\dot{X}(t) = F\cos(t+\Phi) + \dot{Y}(t)$$

$$p_{X\dot{X}}(x,\dot{x}\,|\,\Phi=\phi) = p_{Y\dot{Y}}(y,\dot{y})\Big|_{\substack{y=x-F\sin(t+\phi)\\ \dot{y}=\dot{x}-F\cos(t+\phi)}}$$

$$= \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2\sigma^2}\left\{\left(x-F\sin(t+\phi)\right)^2+\left(\dot{x}-F\cos(t+\phi)\right)^2\right\}\right]$$

$$\Rightarrow p_{X\dot{X}}(x,\dot{x}) = \int_{-\pi}^{\pi} p_{X\dot{X}}(x,\dot{x}\,|\,\Phi=\phi)\,p(\phi)\,d\phi$$

$$= \frac{1}{2\pi\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[-\frac{1}{2\sigma^2}\left\{\left(x-F\sin(t+\phi)\right)^2+\left(\dot{x}-F\cos(t+\phi)\right)^2\right\}\right]d\phi$$

$$= \frac{1}{2\pi\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[-\frac{1}{2\sigma^2}\left\{\left(x-F\sin\psi\right)^2+\left(\dot{x}-F\cos\psi\right)^2\right\}\right]d\psi$$

$$
\begin{aligned}
&p_{X\dot{X}}\left(x,\dot{x}\right)\\
&=\frac{1}{2\pi\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[-\frac{1}{2\sigma^2}\left\{\left(x-F\sin\psi\right)^2+\left(\dot{x}-F\cos\psi\right)^2\right\}\right]d\psi\\
&=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2\sigma^2}\left\{x^2+\dot{x}^2+F^2\right\}\right]\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[-\frac{F}{\sigma^2}\left\{x\sin\psi+\dot{x}\cos\psi\right\}\right]d\psi\\
&=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2\sigma^2}\left\{x^2+\dot{x}^2+F^2\right\}\right]I_0\left[\frac{F}{\sigma^2}\sqrt{x^2+\dot{x}^2}\right];-\infty<x,\dot{x}<\infty
\end{aligned}
$$

$$X\left(t\right)=F\sin\left(t+\Phi\right)+Y\left(t\right)$$

$$\Rightarrow p_X\left(x\,|\,\Phi=\phi\right)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2\sigma^2}\left\{x-F\sin\left(t+\phi\right)\right\}^2\right]$$

$$p_X\left(x\right)=\int_{-\pi}^{\pi}p_X\left(x\,|\,\Phi=\phi\right)p\left(\phi\right)d\phi \quad //$$

$$p_{\dot{X}}\left(\dot{x}\right)=\int_{-\pi}^{\pi}p_X\left(\dot{x}\,|\,\Phi=\phi\right)p\left(\phi\right)d\phi \quad //$$

It can be verified that

$$p_{X\dot{X}}\left(x,\dot{x}\right)\neq p_X\left(x\right)p_{\dot{X}}\left(\dot{x}\right)$$

Remark

- $X\left(t\right)$ and $\dot{X}\left(t\right)$ are uncorrelated because they are stationary random processes.
- $X\left(t\right)$ and $\dot{X}\left(t\right)$ are not independent

Problem 29

Let $X(t)$ be a random process with $\langle X(t) \rangle = \mu_X(t)$ and $\left\langle \left[X(t) - \mu_X(t) \right]^2 \right\rangle = \sigma_X^2(t)$. Show that

$$\mathrm{P}\left[\left| X(t) - \mu_X(t) \right| \geq \varepsilon \text{ for some } t \text{ in } a \leq t \leq b \right]$$

$$\leq \frac{1}{2\varepsilon^2}\left[\sigma_X^2(a) + \sigma_X^2(b) \right] + \frac{1}{\varepsilon^2}\int_a^b \sigma_X(t)\,\sigma_{\dot{X}}(t)\,dt; \varepsilon > 0$$

Note: This is the generalization of Chebychev's inequality for random processes.

[Hints] We have for a random process $Y(t)$

$$\mathrm{P}\left[\sup_{a\le t\le b}\left|Y(t)\right|\ge\varepsilon\right]\le\frac{1}{\varepsilon^{2}}\mathrm{E}\left[\sup_{a\le t\le b}Y^{2}(t)\right]\ \text{[Prove this]}$$

Also,

$$Y^{2}(t)=Y^{2}(a)+2\int_{a}^{t}\left[\frac{d}{du}Y(u)\right]Y(u)\,du$$

$$=Y^{2}(b)-2\int_{t}^{b}\left[\frac{d}{du}Y(u)\right]Y(u)\,du$$

$$\Rightarrow Y^{2}(t)\le\frac{1}{2}\left[Y^{2}(a)+Y^{2}(b)\right]+\int_{a}^{t}\left|\left[\frac{d}{du}Y(u)\right]Y(u)\right|du$$

$$\Rightarrow\sup_{a\le t\le b}Y^{2}(t)\le\frac{1}{2}\left[Y^{2}(a)+Y^{2}(b)\right]+\int_{a}^{t}\left|\left[\frac{d}{du}Y(u)\right]Y(u)\right|du$$

We have

$$\mathrm{E}\left[\left|UV\right|\right] \le \left\{\mathrm{E}\left[U^2\right]\mathrm{E}\left[V^2\right]\right\}^{\frac{1}{2}}$$

$\Rightarrow$

$$E\left[\sup_{a\le t\le b} Y^2(t)\right] \le \frac{1}{2}E\left[Y^2(a)+Y^2(b)\right]$$

$$+\int_a^t\left\{\mathrm{E}\left\{\left[\frac{d}{du}Y(u)\right]^2\right\}\mathrm{E}\left[Y^2(u)\right]\right\}^{\frac{1}{2}}du$$

Substitute $Y(t) = X(t) - \mu_X(t)$ in the above to get the required result.

Problem 30

Let $X(t)$ be a stationary random process with zero mean and autocovariance function given by

$$R_{XX}(\tau) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\tau^2}{2\sigma^2} \right) //$$

- How many times can we differentiate this process?
- Determine $P\left[\dot{X}(t) \leq 0.75 \right]$ if it is given that the process is Gaussian and σ=1.

$$R_{XX}(\tau)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\tau^2}{2\sigma^2}\right)$$

$$\Rightarrow S_{XX}(\omega)=\exp\left(-\frac{\sigma^2\omega^2}{2}\right)$$

$$\lambda_n=\int_{-\infty}^{\infty}\omega^n\exp\left(-\frac{\sigma^2\omega^2}{2}\right)d\omega \text{ exist } \forall n=1,2,\cdots$$

$\Rightarrow R_{XX}(\tau)$ is differentiable at $\tau=0$ for all orders.

$\Rightarrow X(t)$ is differentiable to any order n

(in the mean square sense)

$$R_{XX}(\tau) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\tau^2}{2\sigma^2} \right)$$

$$\frac{d}{d\tau} R_{XX}(\tau) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\tau^2}{2\sigma^2} \right) \left(-\frac{\tau}{\sigma^2} \right)$$

$$\frac{d^2}{d\tau^2} R_{XX}(\tau) = \left(-\frac{1}{\sigma^2} \right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\tau^2}{2\sigma^2} \right)$$

$$+\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\tau^2}{2\sigma^2} \right) \left(-\frac{\tau}{\sigma^2} \right)^2$$

$$\Rightarrow R_{\dot{X}\dot{X}}(\tau = 0) = \left(\frac{1}{\sigma^2} \right) \frac{1}{\sqrt{2\pi}\sigma} = \frac{1}{\sqrt{2\pi}} \because \sigma = 1$$

$$p_{\dot{X}}(\dot{x}) = \exp\left(-\frac{1}{2} 2\pi \dot{x}^2 \right) = \exp\left(-\pi \dot{x}^2 \right)$$

$$P\left[\dot{X}(t) \le 0.75 \right] = \int_{-\infty}^{0.75} \exp\left(-\pi \dot{x}^2 \right) d\dot{x} = 0.97$$

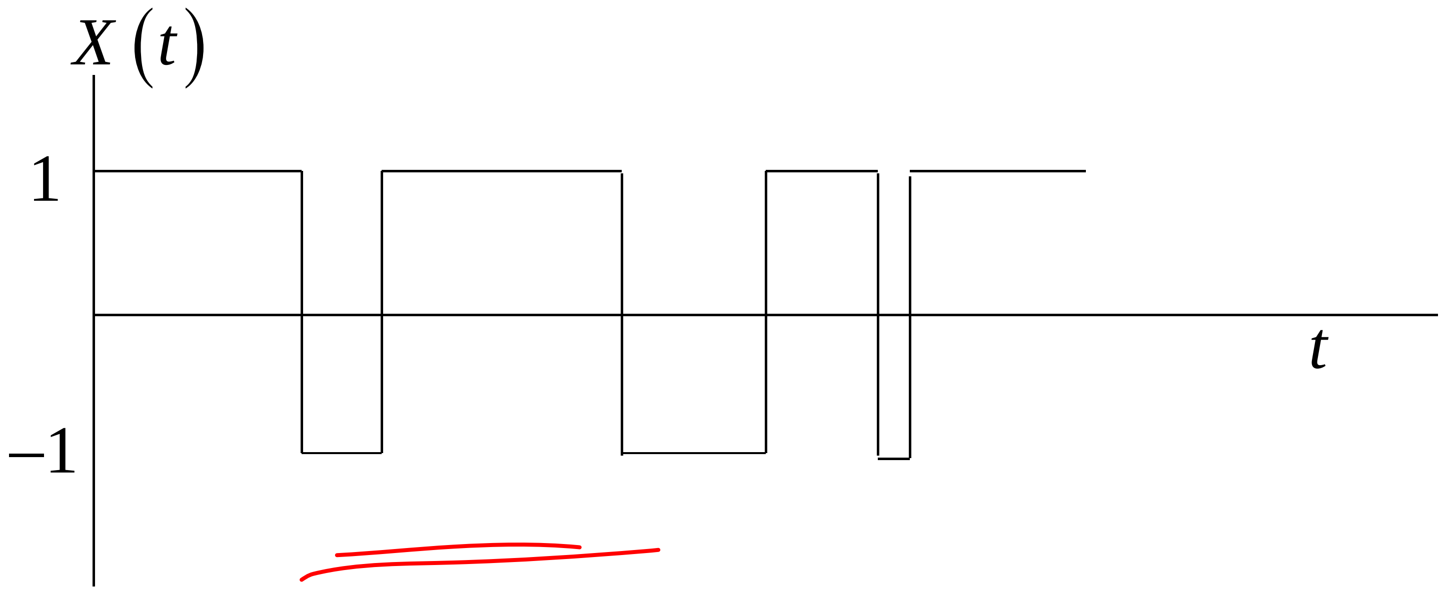
Problem 31

Let $N(t)$ be a Poisson random process with arrival rate λ.

Define $X(t) = (-1)^{N(t)}$.

Determine the mean and covariance of $X(t)$.

Note: $X(t)$ is known as semi random telegraph signal.



$$X(t) = (-1)^{N(t)}.$$

$$\Rightarrow X(t) = \begin{cases} 1 \text{ if } N(t) = 0 \text{ or } N(t) \text{ is even} \\ -1 \text{ if } N(t) \text{ is odd} \end{cases}$$

$$P\left[X(t) = 1\right] = P\left[N(t) = 0 \text{ or } N(t) \text{ is even}\right]$$

$$= \exp(-\lambda t)\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] = \exp(-\lambda t)\cosh \lambda t$$

$$P\left[X(t) = -1\right] = P\left[\, N(t) \text{ is odd}\right]$$

$$= \exp(-\lambda t)\left[\lambda t + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots\right] = \exp(-\lambda t)\sinh \lambda t$$

$$\langle X(t) \rangle = P\left[X(t) = 1\right](1) + P\left[X(t) = -1\right](-1)$$

$$= \exp(-\lambda t)\left[\cosh \lambda t - \sinh \lambda t\right] = 2\exp(-2\lambda t)$$

$X(t)X(t+\tau) = 1$ if there are even number of occurrences in t to $t+\tau$.

$X(t)X(t+\tau) = -1$ if there are odd number of occurrences in t to $t+\tau$.

$\Rightarrow$

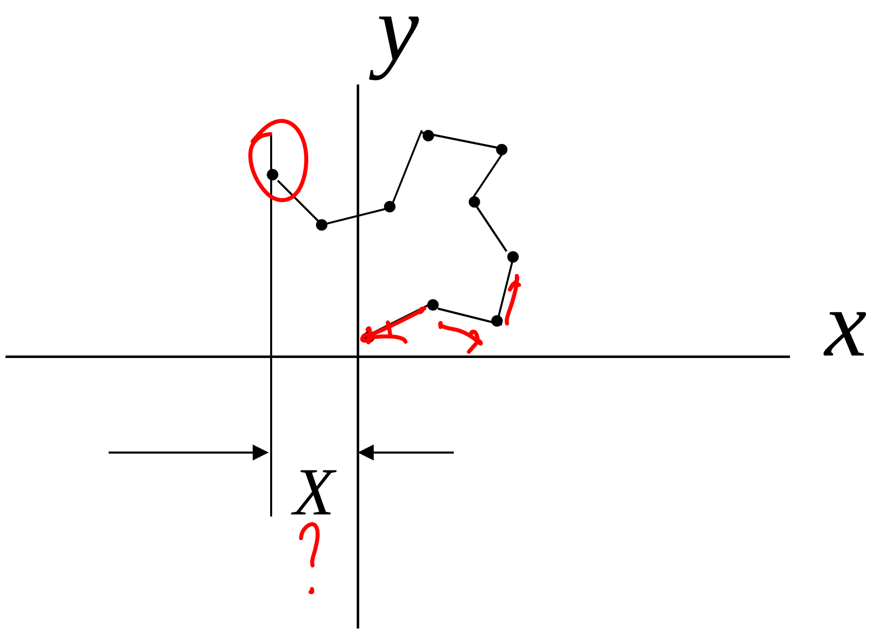
$$\langle X(t)X(t+\tau)\rangle =$$

$$(1)\sum_{n \text{ even}} \exp(-\lambda\tau)\frac{(\lambda\tau)^n}{n!} + (-1)\sum_{n \text{ odd}} \exp(-\lambda\tau)\frac{(\lambda\tau)^n}{n!} = \exp(-2\lambda\tau)$$

$$\Rightarrow R_{XX}(t, t+\tau) = R_{XX}(\tau) = \exp(-2\lambda|\tau|)$$

Problem 32

A random walk is performed on a two-dimensional plane with a uniform step size of Δ. At every step the direction α_i is a random variable. α_i-s can be taken to be an iid sequence with a common pdf that is uniformly distributed in 0 to 2π. Find the distribution of the x-coordinate after n steps.



$$X = \sum_{i=1}^{N} \Delta \cos \alpha_i$$

$$\Phi_X(\omega) = \left\langle \exp\left(i\omega X\right)\right\rangle = \left\langle \exp\left(i\omega \sum_{j=1}^{N} \Delta \cos \alpha_j\right)\right\rangle$$

$$= \prod_{i=1}^{n} \left\langle \exp\left(i\omega\Delta \cos \alpha_j\right)\right\rangle$$

$$\left\langle \exp\left(i\omega\Delta \cos \alpha_j\right)\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(i\omega\Delta \cos \alpha_j\right) d\alpha_j = J_0\left(\omega\Delta\right)$$

$$\Rightarrow \Phi_X(\omega) = \left[J_0\left(\omega\Delta\right)\right]^n$$

$$p_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[J_0\left(\omega\Delta\right)\right]^n \exp\left(-i\omega x\right) d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty} \left[J_0\left(\omega\Delta\right)\right]^n \cos \omega x d\omega$$

$$\Phi_X(\omega) = \left[J_0(\omega\Delta) \right]^n$$

$$p_X(x) = \frac{1}{\pi} \int_0^\infty \left[J_0(\omega\Delta) \right]^n \cos \omega x d\omega$$

$$\left[J_0(\omega\Delta) \right]^n = \left(1 - \frac{\Delta^2 \omega^2}{2^2} + \frac{\Delta^4 \omega^4}{2^2 4^2} - \cdots \right)^n$$

Consider the limit $n \to \infty$ such that $\Delta\sqrt{n} \to c$.

$$\left[J_0(\omega\Delta) \right]^n = \left(1 - \frac{c^2 \omega^2}{n 2^2} + \frac{c^4 \omega^4}{n^2 2^2 4^2} - \cdots \right)^n$$

$$= \left(1 - \frac{c^2 \omega^2}{n 2^2} \right)^n = \exp\left(-\frac{c^2 \omega^2}{4} \right)$$

$$p_X(x) = \frac{1}{\sqrt{\pi} c} \exp\left(-\frac{x^2}{c^2} \right); -\infty < x < \infty$$

Problem 33

Let the time interval 0 to T be divided into a sequence of equal intervals of length T. Consider a sequence of n Bernoulli trials with $\mathrm{P}(\text{success}) = \frac{1}{2}$. Define

$$X(t) = \begin{cases} 1 \text{ if success on } n^{\text{th}} \text{ trial} \\ -1 \text{ if failure on } n^{\text{th}} \text{ trial} \end{cases} (n-1)T < t < nT$$

Find mean and autocorrelation of $X(t)$.

Furtheremore, let e be a random variable distributed uniformly in 0 to T and independent of $X(t)$.

Define $Y(t) = X(t - e)$. Determine the mean and autocorrelaiton of $Y(t)$.

$$X(t) = \begin{Bmatrix} 1 \text{ if success on } n^{\text{th}} \text{ trial} \\ -1 \text{ if failure on } n^{\text{th}} \text{ trial} \end{Bmatrix} (n-1)T < t < nT$$

$$\langle X(t) \rangle = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$$

$$\langle X^2(t) \rangle = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$$

$$\langle X(t_1) X(t_2) \rangle = \begin{cases} 1 \text{ if } (n-1)T < t_1, t_2 < nT \\ \quad 0 \text{ otherwise} \end{cases}$$

$$Y(t) = X(t-\varepsilon)$$

$$\Rightarrow \mathrm{E}\left[Y(t)\right] = \mathrm{E}\left[\mathrm{E}\left[Y(t)\middle|\varepsilon\right]\right] = \mathrm{E}\left[X(t-\varepsilon)\middle|\varepsilon\right] = 0$$

$$\mathrm{E}\left[Y(t_1)Y(t_2)\right] = \mathrm{E}\left[X(t_1-\varepsilon)X(t_1-\varepsilon)\right]$$

$$= \mathrm{E}\left[\mathrm{E}\left[X(t_1-\varepsilon)X(t_2-\varepsilon)\middle|\varepsilon\right]\right]$$

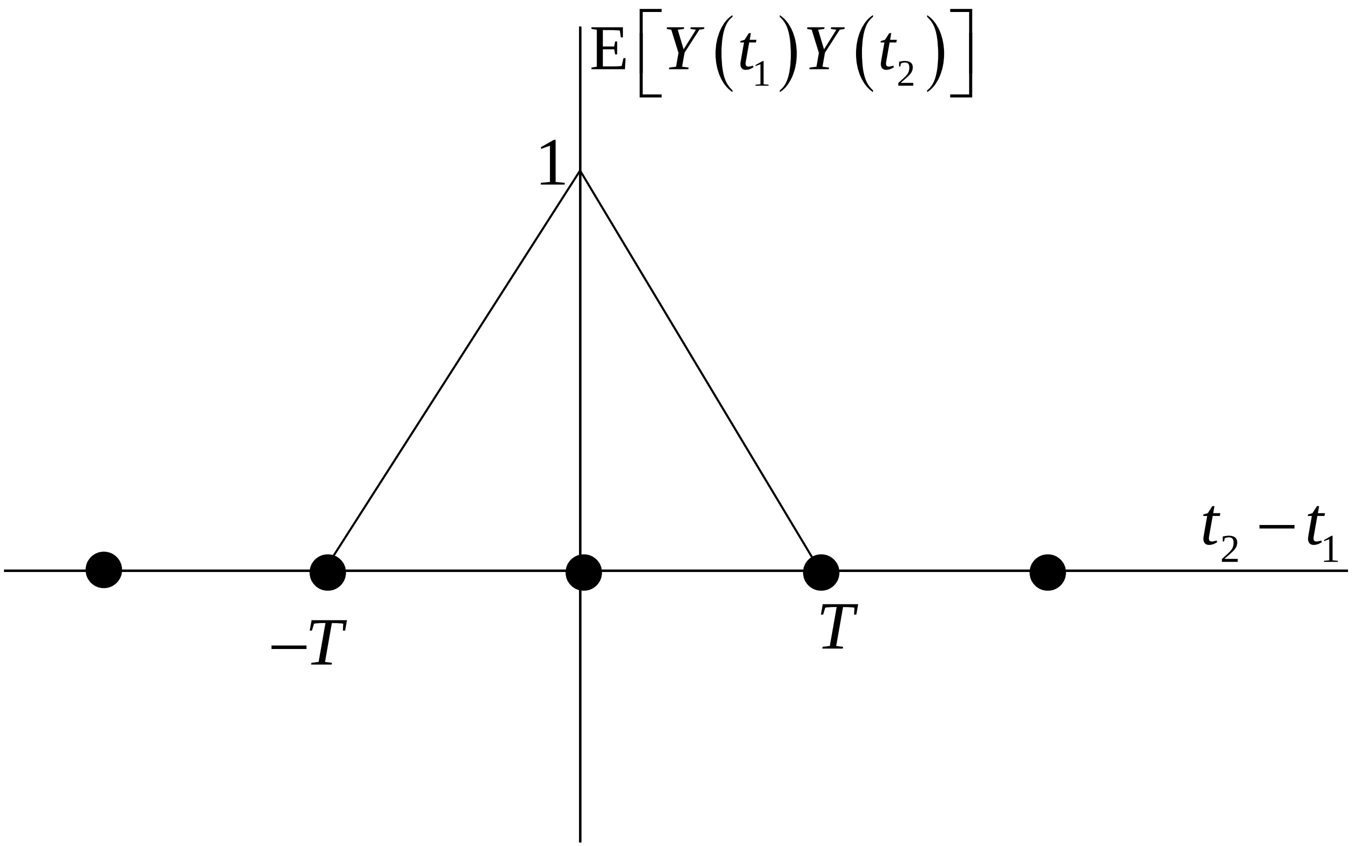
$$\mathrm{E}\left[X(t_1-\varepsilon)X(t_2-\varepsilon)\middle|\varepsilon\right] = 0 \text{ if } \left|t_1-t_2\right| > T$$

$$\text{If } \left|t_1-t_2\right| < T$$

$$\mathrm{E}\left[X(t_1-\varepsilon)X(t_2-\varepsilon)\middle|\varepsilon\right] = \begin{cases} 1 \text{ if } e < T - \left|t_1-t_2\right| \\ 0 \text{ otherwise} \end{cases}$$

$$\Rightarrow \mathrm{E}\left[Y(t_1)Y(t_2)\right] = 1 \times P\left[e < T - \left|t_1 - t_2\right|\right]$$

$$= 1 - \frac{\left|t_1 - t_2\right|}{T}$$



Problem 34

Given a postive function $S(\omega)$ find a stochastic process whose PSD is $S(\omega)$. $\left[\text{Existence theorem}\right]$

Determine a LTI system with $\mathrm{H}\left(i\omega\right)=\sqrt{S\left(\omega\right)}$ and pass a zero mean stationary Gaussian white noise with unit strength through this system. The output process would be a zero mean stationary process with PSD=$S\left(\omega\right)$.

Alternative

Determine $a^2 = \int_{-\infty}^{\infty} S(\omega)\,d\omega$ and define $f(\omega) = \dfrac{S(\omega)}{a^2}$.

Clearly, $f(\omega) \geq 0 \;\&\; \int_{-\infty}^{\infty} f(\omega)\,d\omega = 1$; also, $f(\omega) = f(-\omega)$

$\Rightarrow$ $f(\omega)$ has the properties of a pdf of a random variable.

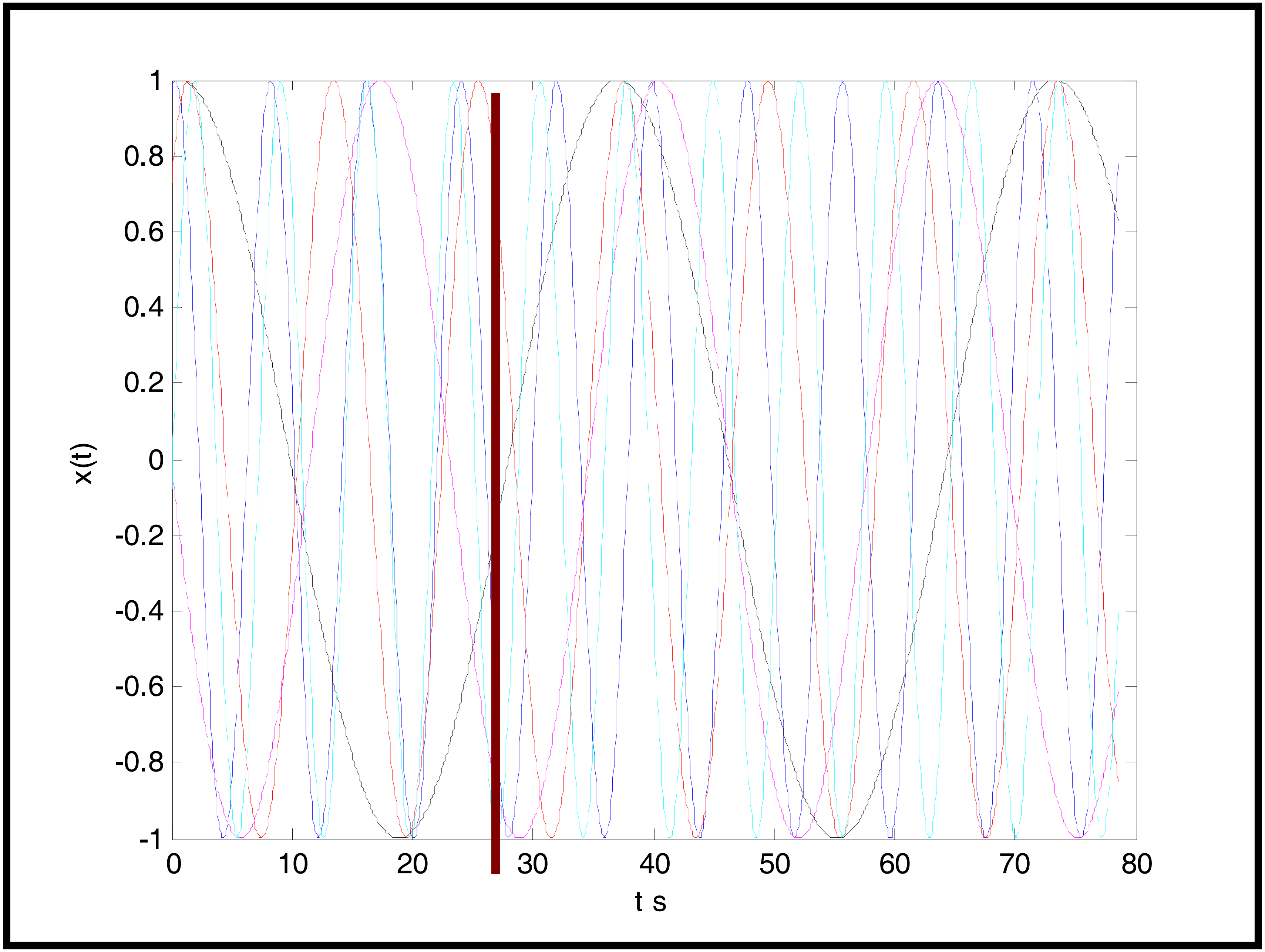
Let $X(t) = a\cos(\omega t + \phi)$ where ω and ϕ are random variables

with $\omega \sim f(\omega), \phi \sim U[0, 2\pi], \& \omega \perp \phi$.

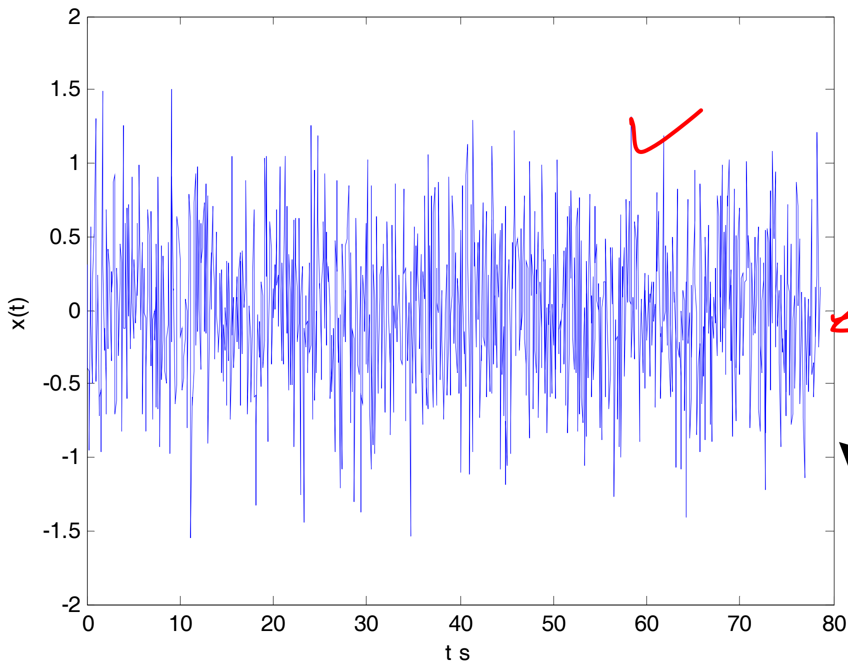
$\langle X(t) \rangle = 0$ [Prove it; start with finding mean conditioned on ω]

$$\langle X(t_1) X(t_2) \rangle = \int_{-\infty}^{\infty} \int_{0}^{2\pi} a\cos(\omega t_1 + \phi)\, a\cos(\omega t_2 + \phi)\, p_{\omega\phi}(\omega, \phi)\, d\omega\, d\phi$$

$$R_{XX}(\tau) = \frac{a^2}{2\pi} \int_{-\infty}^{\infty} \cos\omega\tau\, f(\omega)\, d\omega \Rightarrow S_{XX}(\omega) = a^2 f(\omega) \quad [\text{OK}]$$



x(t)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
60
70
80
t s

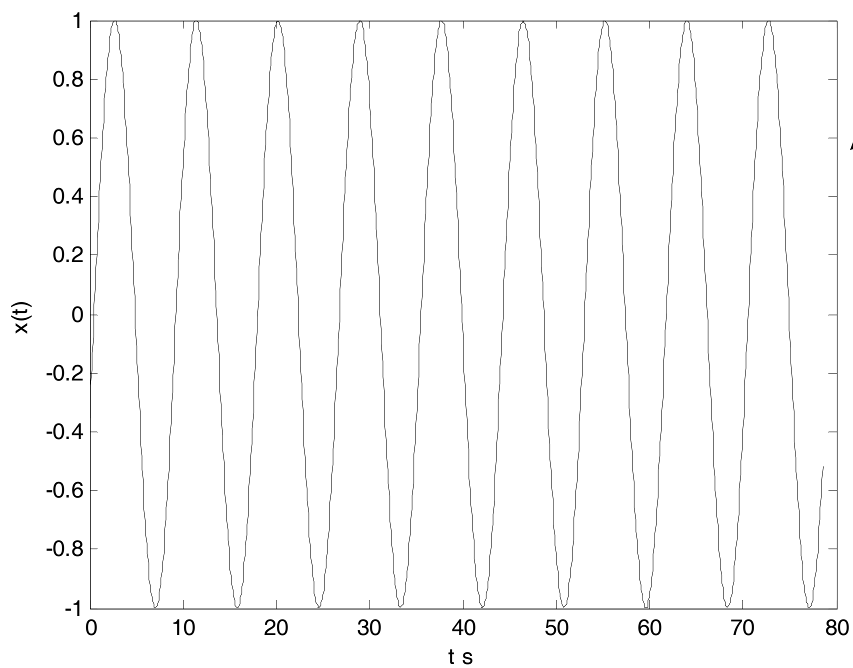


$\dot{x} + \alpha x = \xi(t)$

$\frac{1}{\alpha^2 + \omega^2}$

Remarks

- PSD is an ensemble property
- These two time histories represent samples from two different processes having the same mean and PSD function.



$x(t) = a\cos(\omega t + \phi)$

Remark

The following three processes share the same form of PSD

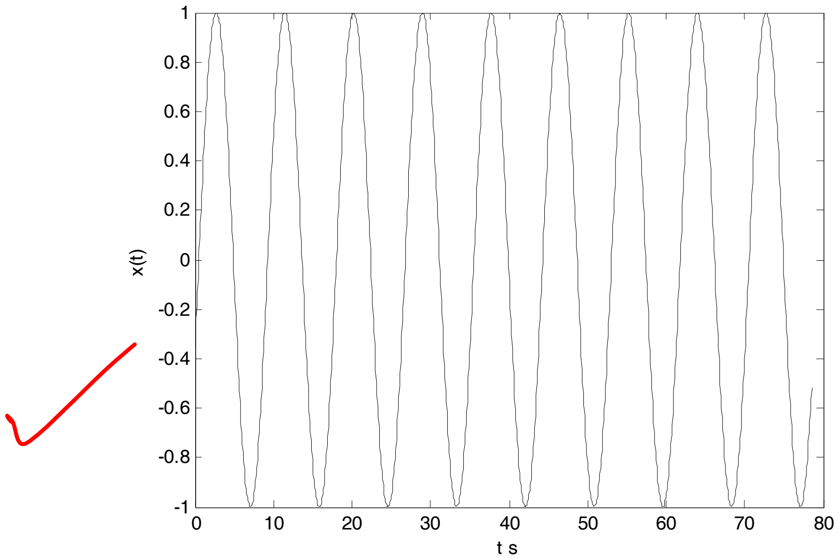
The sample realizations are dramatically different.

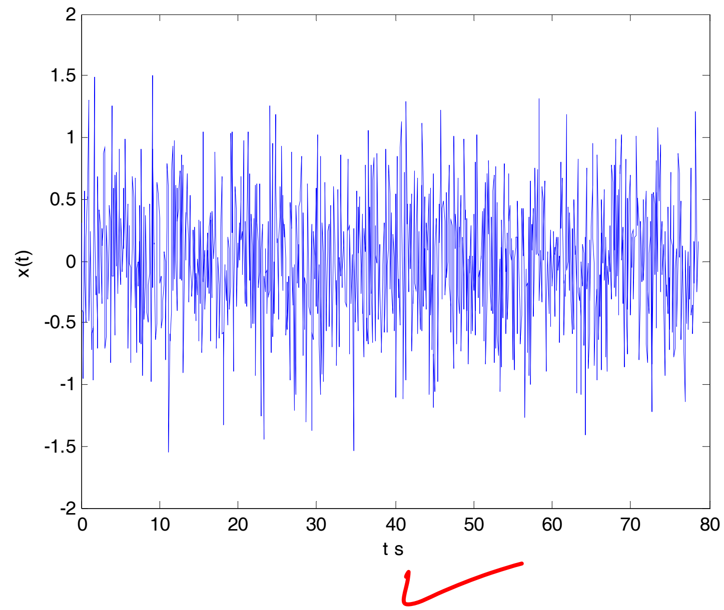
$\dot{X}(t) - m_X(t)$

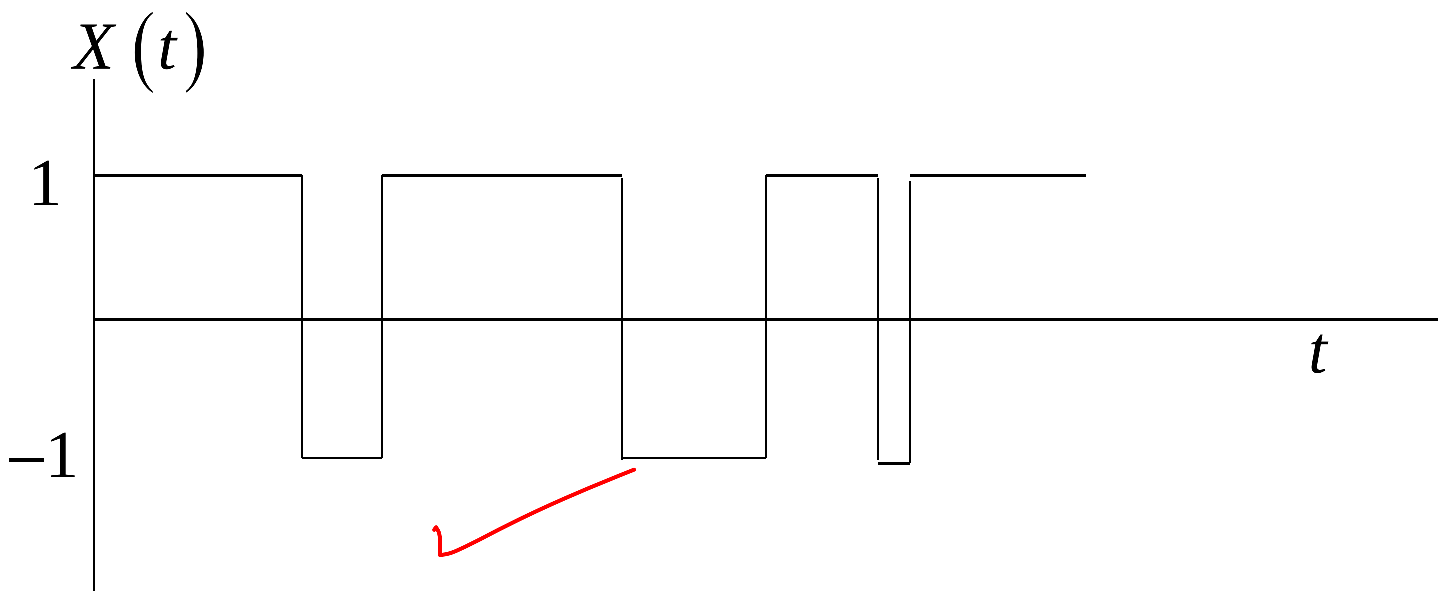
- $X(t) = (-1)^{N(t)}$; $N(t)$: Poisson process with rate λ

- $X(t) = \cos(\omega t + \phi); \omega \sim \dfrac{S(\omega)}{\int_{-\infty}^{\infty} S(\omega)\, d\omega}; \phi \sim U[0, 2\pi]; \omega \perp \phi$

- [Steady state response] $\dot{X} + \alpha X = \xi(t); X(0) = X_0$


x(t)
t s


x(t)
t s


$X(t)$
1
−1
t

Discussion on properties of processes with Independent increments

Factor of safety
& Prob of failure