Stochastic Structural Dynamics

Lecture-37

Problem solving session-1

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Problem 1

A coin is tossed 10 times. Probability of obtaining head on any trial is given to be 0.4. Find the probability that head shows up no less than 5 times and no more than 3 times.

- Repeated Bernoulli trials and binomial distribution
- $P\left(X=k\right)={}^{n}C_{k}\,p^{k}\left(1-p\right)^{n-k};k=1,2,\cdots,n$

$n=10;\ p=0.4;$

$P\left(\text{head shows up no less than 5 times and no more than 3 times}\right)$

$$=P\left(3\leq X\leq 5\right)=\sum_{k=3}^{5}{}^{10}C_{k}\left(0.4\right)^{k}\left(1-0.4\right)^{10-k}$$

$$={}^{10}C_{3}\left(0.4\right)^{3}\left(0.6\right)^{7}+{}^{10}C_{4}\left(0.4\right)^{4}\left(0.6\right)^{6}+{}^{10}C_{5}\left(0.4\right)^{5}\left(0.6\right)^{0.5}$$

$$=0.0425+0.1115+0.2007=0.3547$$

Problem 2

Let X be a Poisson random variable with

$\mathrm{P}\left[X = k\right] = \exp\left(-a\right)\dfrac{a^k}{k!}$ with parameter $a = 1.0$.

Define a new random variable $Y = \min(X, 4)$. Determine the characteristic function of Y.

Y assumes five distinct values: 0,1,2,3,4

$$P\left[Y = 0\right] = P\left[X = 0\right] = \exp\left(-1\right) = 0.3679$$

$$P\left[Y = 1\right] = P\left[X = 1\right] = \exp\left(-1\right) = 0.3679$$

$$P\left[Y = 2\right] = P\left[X = 2\right] = \exp\left(-1\right)\frac{1}{2!} = 0.1839$$

$$P\left[Y = 3\right] = P\left[X = 3\right] = \exp\left(-1\right)\frac{1}{3!} = 0.0613$$

$$P\left[Y = 4\right] = 1 - P\left[X \le 3\right] = 1 - 0.9810 = 0.0190$$

$$\Phi_Y(\omega) = \left\langle \exp\left[i\omega Y\right] \right\rangle$$

$$= \sum_{k=0}^{4} P\left[Y = k\right] \exp\left[i\omega k\right]$$

$$= 0.3679 + 0.3679 \exp\left[i\omega\right] + 0.1839 \exp\left[2i\omega\right] +$$

$$0.0613 \exp\left[3i\omega\right] + 0.0190 \exp\left[4i\omega\right]$$

Problem 3

Let X be a normal random variable with paramters m and σ.

Show that

$$\langle X \rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx = m$$

$$\left\langle (X - m)^2 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-m)^2 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx = \sigma^2$$

$$\phi_X(\omega) = \exp\left(im\omega - \frac{1}{2}\sigma^2\omega^2\right)$$

$$\langle X \rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma} \right)^2 \right] dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (m + x - m) \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma} \right)^2 \right] dx$$

$$= \frac{m}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma} \right)^2 \right] dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{x-m}{\sigma} \right) \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma} \right)^2 \right] dx$$

$$\text{Put } \frac{x-m}{\sigma} = u \Rightarrow dx = \sigma du$$

$$= m + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \exp\left(-\frac{u^2}{2} \right) \sigma du = m$$

$$\left\langle (X - m)^2 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-m)^2 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx$$

We have

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = \sqrt{2\pi}\sigma$$

Differentiate with respect to $\sigma \Rightarrow$

$$\int_{-\infty}^{\infty} \frac{(x-m)^2}{\sigma^3} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = \sqrt{2\pi}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-m)^2 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = \sigma^2$$

To show that $\phi_X(\omega) = \exp\left(im\omega - \frac{1}{2}\sigma^2\omega^2 \right)$

$$\phi_X(\omega) = \left\langle \exp\left(i\omega X\right)\right\rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left(i\omega x\right)\exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]dx$$

$$u = \frac{x-m}{\sigma} \Rightarrow \sigma du = dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left(i\omega\left(\sigma u + m\right)\right)\exp\left[-\frac{1}{2}u^2\right]\sigma du$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(im\omega\right)\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(u^2 - 2i\omega u\sigma + \left(i\sigma\omega\right)^2 - \left(i\sigma\omega\right)^2\right)\right]du$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(im\omega\right)\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(u - im\omega\right)^2 + \frac{1}{2}\sigma^2\omega^2\right]du$$

$$= \exp\left(im\omega - \frac{1}{2}\sigma^2\omega^2 \right)$$

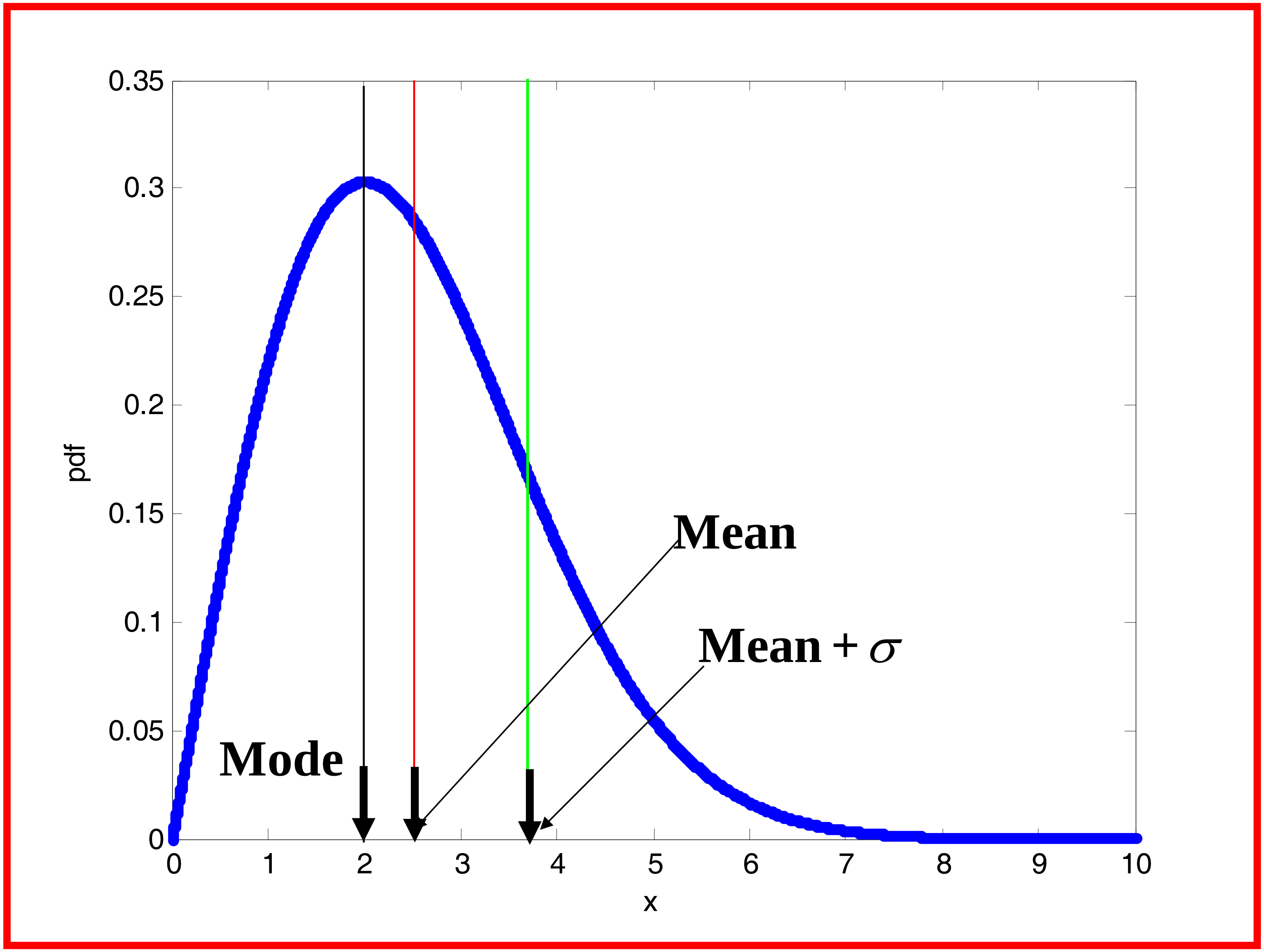
Problem 4

Let X be Rayleigh random variable with pdf given by

$$p_X(x) = \frac{x}{4}\exp\left(-\frac{x^2}{8}\right); x \geq 0.$$

Determine

- $\mathrm{P}\left[X > \mathrm{Mode}\right]$
- $P\left[X > \mathrm{Mean}\right]$
- $P\left[X > \mathrm{Mean+standard\ deviation}\right]$



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8
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10
x
Mean
Mean + σ
Mode

$$p_X(x) = \frac{x}{4}\exp\left(-\frac{x^2}{8}\right); x \geq 0$$

Is this a valid pdf?

Area under pdf= $A = \int_0^\infty \frac{x}{4}\exp\left(-\frac{x^2}{8}\right)dx$

$$t = \frac{x^2}{8} \Rightarrow dt = \frac{x}{4}dx \Rightarrow A = \int_0^\infty \exp(-t)\,dt = 1 \text{ ok}$$

$$P_X(x) = \int_0^x \frac{u}{4}\exp\left(-\frac{u^2}{8}\right)du = \int_0^{\frac{x^2}{8}} \exp(-t)\,dt = 1 - \exp\left(-\frac{x^2}{8}\right)$$

Mode: x^* such that $\frac{dp_X(x)}{dx} = 0$ at $x = x^*$.

$$\frac{dp_X(x)}{dx} = \frac{1}{4}\exp\left(-\frac{x^2}{8}\right) + \frac{x}{4}\exp\left(-\frac{x^2}{8}\right)\left(-\frac{x}{4}\right) = 0 \Rightarrow x = 2$$

$$P\left[X > \text{Mode}\right] = P\left[X > 2\right] = 1 - P\left[X \le 2\right]$$

$$= 1 - \left\{1 - \exp\left(-\frac{2^2}{8}\right)\right\} = \exp\left(-\frac{1}{2}\right) = 0.6065$$

$$\langle X \rangle = \int_0^\infty \frac{x^2}{4} \exp\left(-\frac{x^2}{8}\right) dx$$

Consider $U \sim N(0,2)$

$$\Rightarrow \langle U^2 \rangle = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}\,2} \exp\left(-\frac{u^2}{8}\right) du = 4$$

$$\Rightarrow \int_0^\infty u^2 \exp\left(-\frac{u^2}{8}\right) du = 4\sqrt{2\pi}$$

$$\langle X \rangle = \int_0^\infty \frac{x^2}{4} \exp\left(-\frac{x^2}{8}\right) dx = \frac{1}{4} 4\sqrt{2\pi} = \sqrt{2\pi}$$

$$\left\langle X^2 \right\rangle = \int_0^\infty \frac{x^3}{4} \exp\left(-\frac{x^2}{8}\right) dx = \int_0^\infty \frac{x}{4} x^2 \exp\left(-\frac{x^2}{8}\right) dx$$

$$\text{Put } \frac{x^2}{8} = t \Rightarrow \frac{x}{4} dx = dt$$

$$\Rightarrow \left\langle X^2 \right\rangle = \int_0^\infty 8t \exp(-t)\, dt = 8$$

$$\sigma_X^2 = 8 - 2\pi = 1.7168 \Rightarrow \sigma_X = 1.3103$$

$$\text{Mode=2} \Rightarrow P(X > 2) = \exp\left(-\frac{4}{8}\right) = 0.6065$$

$$\text{Mean=}\sqrt{2\pi} = 2.5066 \Rightarrow P(X > \text{Mean}) = 0.4559$$

$$\text{Mean+std dev=}2.5066 + 1.3103 = 3.8169$$

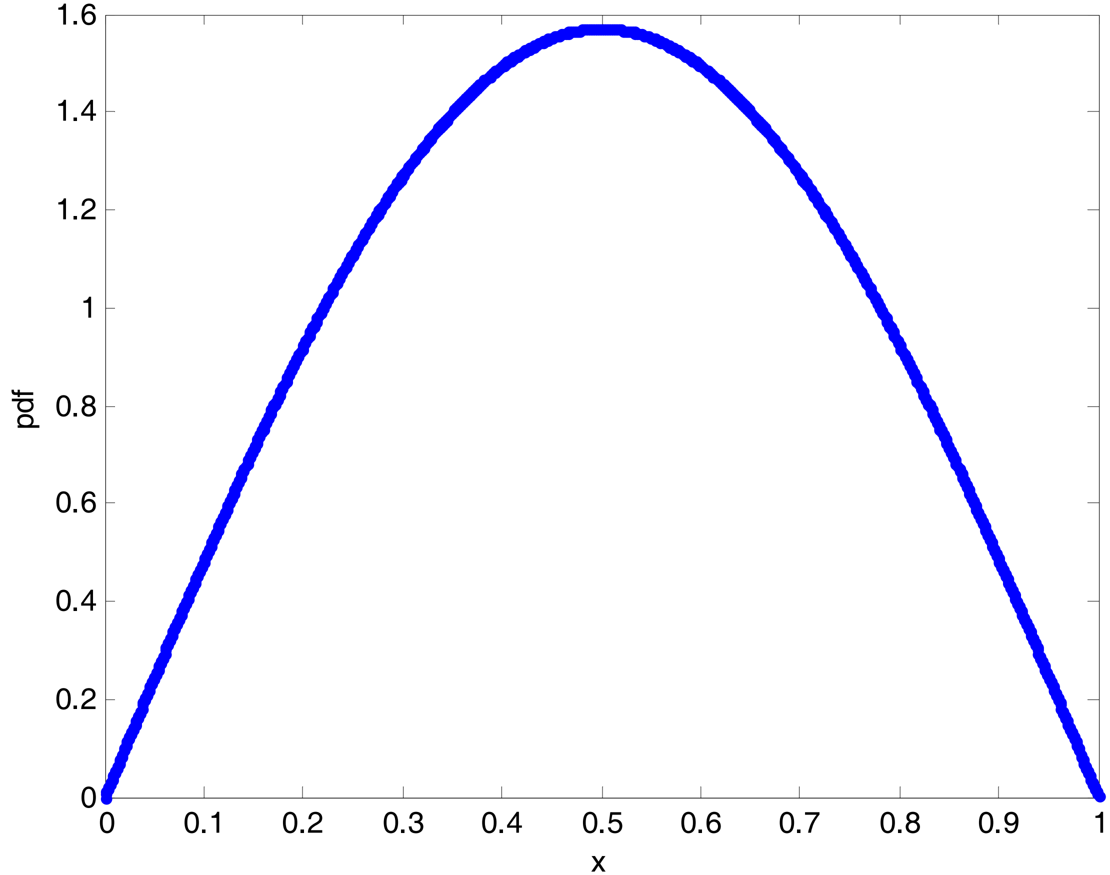
$$P(X > \text{Mean+std dev}) = 0.1618$$

Problem 5

The pdf of a random variable X is given by

$p_X(x) = A\sin \pi x; 0 < x < 1.$

- Determine A
- Define $Y = \cos \pi X$. Determine mean and variance of Y.
- Repeat the exercise if Y is now defined as $Y = \left|\cos \pi X\right|$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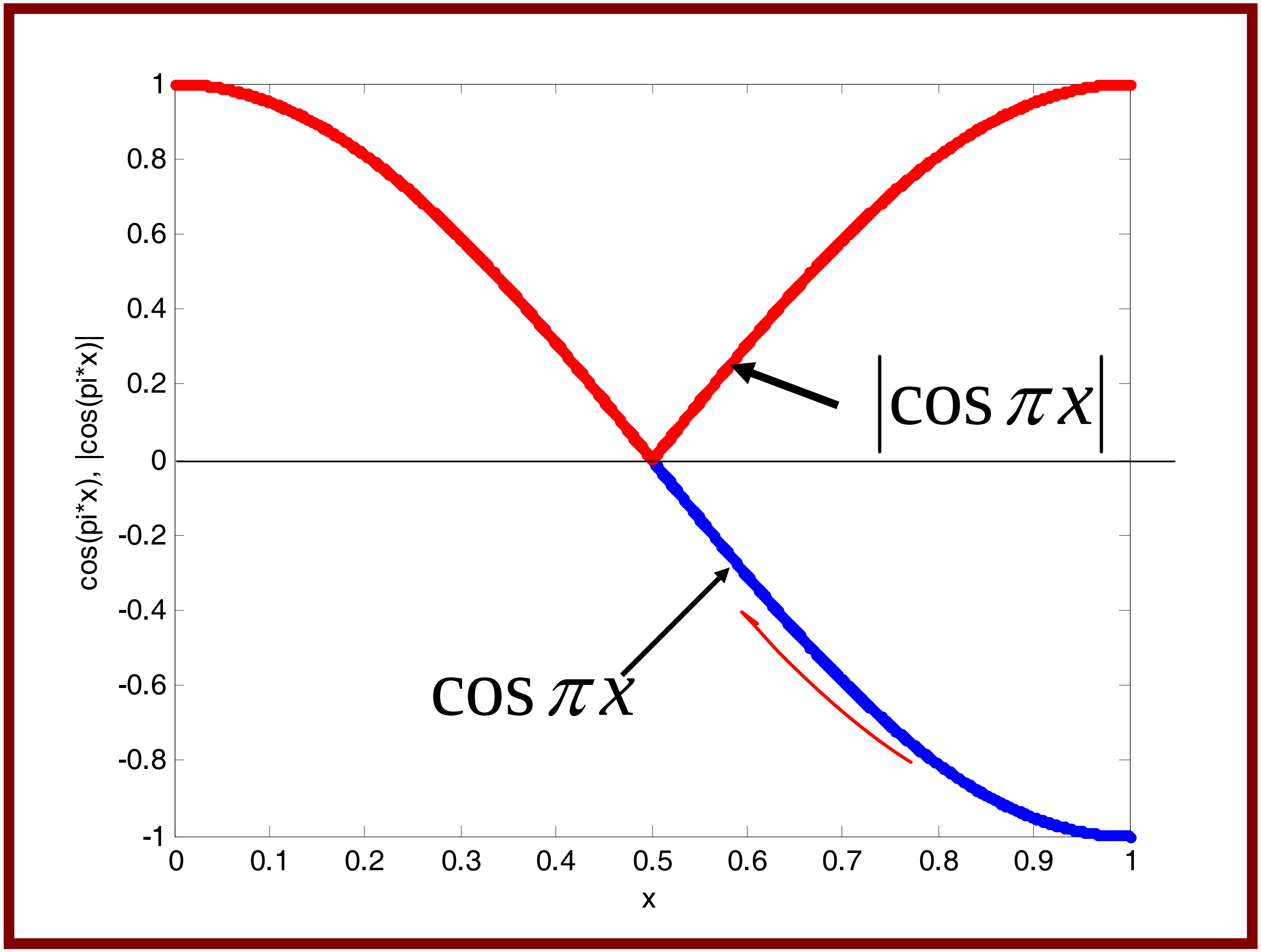
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-0.8
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0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
cos(pi*x), |cos(pi*x)|
$|\cos \pi x|$
$\cos \pi x$

$$\int_0^1 A\sin\pi x dx = 1 \Rightarrow A = \frac{\pi}{2}$$

$$\Rightarrow p_X(x) = \frac{\pi}{2}\sin\pi x; 0 \le x \le 1$$

$$Y = \cos\pi X$$

$$\langle Y \rangle = \int_0^1 \cos\pi X \frac{\pi}{2}\sin\pi x dx$$

$$= \frac{\pi}{2}\frac{1}{2}\int_0^1 \sin 2\pi x dx = \frac{\pi}{4}\left(-\frac{\cos 2\pi x}{2\pi}\right)_0^1 = 0$$

$$\langle Y^2 \rangle = \int_0^1 \cos^2\pi X \frac{\pi}{2}\sin\pi x dx$$

Substitute $\cos\pi x = t \Rightarrow -\pi\sin\pi x dx = dt$

$$\Rightarrow \langle Y^2 \rangle = \frac{\pi}{2}\int_1^{-1} t^2\left(-\frac{dt}{\pi}\right) = \frac{1}{2}\times 2\times\frac{1}{3} = \frac{1}{3}$$

$$Y = \left|\cos\pi X\right| \Rightarrow \langle Y\rangle = \int_0^1 \left|\cos\pi x\right| \frac{\pi}{2}\sin\pi x dx$$

$$= \int_0^{\frac{1}{2}} \cos\pi x \frac{\pi}{2}\sin\pi x dx - \int_{\frac{1}{2}}^1 \cos\pi x \frac{\pi}{2}\sin\pi x dx$$

$$= \frac{\pi}{2}\frac{1}{2}\int_0^{\frac{1}{2}} \sin 2\pi x dx - \frac{\pi}{2}\frac{1}{2}\int_{\frac{1}{2}}^1 \sin 2\pi x dx$$

$$= \frac{\pi}{4}\left(-\frac{\cos 2\pi x}{2\pi}\right)_0^{\frac{1}{2}} - \frac{\pi}{4}\left(-\frac{\cos 2\pi x}{2\pi}\right)_{\frac{1}{2}}^1 = \frac{1}{8}(1+1) - \frac{1}{8}(-1+1) = \frac{1}{4}$$

$$\langle Y^2\rangle = \int_0^1 \cos^2\pi X \frac{\pi}{2}\sin\pi x dx = \frac{1}{3}$$

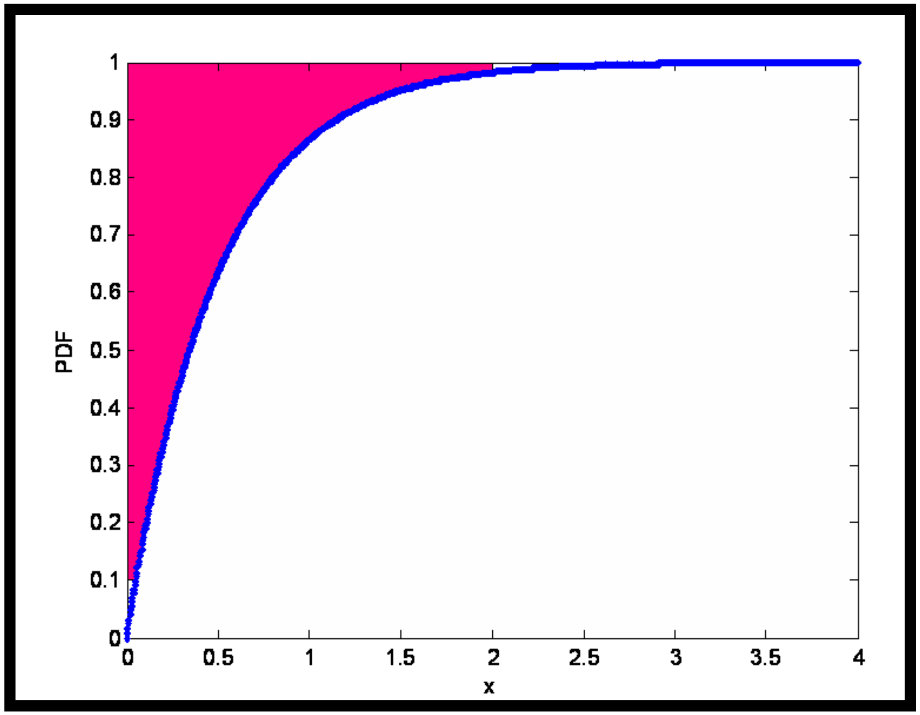
$$\sigma_Y^2 = \frac{1}{3} - \left(\frac{1}{4}\right)^2 = \frac{13}{48} = 0.2708$$

Problem 6

Let X be a random variable with $P_X(x) = 1 - \exp(-\lambda x); x \geq 0$

•Determine the characteristic function and hence determine the mean and standard deviation of X.

•Show that the shaded area (red) in the figure is equal to the mean of the random variable.



$$P_X(x) = 1 - \exp(-\lambda x); x \geq 0$$

$$p_X(x) = \frac{dP_X(x)}{dx} = \lambda \exp(-\lambda x); x \geq 0$$

$$\Phi_X(\omega) = \langle \exp(i\omega X) \rangle = \int_0^\infty \exp(i\omega x)\lambda \exp(-\lambda x)\,dx$$

$$= \lambda \int_0^\infty \exp\left[-(\lambda - i\omega)x\right]dx$$

$$= \lambda \left[\frac{\exp\left[-(\lambda - i\omega)x\right]}{-(\lambda - i\omega)} \right]_0^\infty = \frac{\lambda}{(\lambda - i\omega)}$$

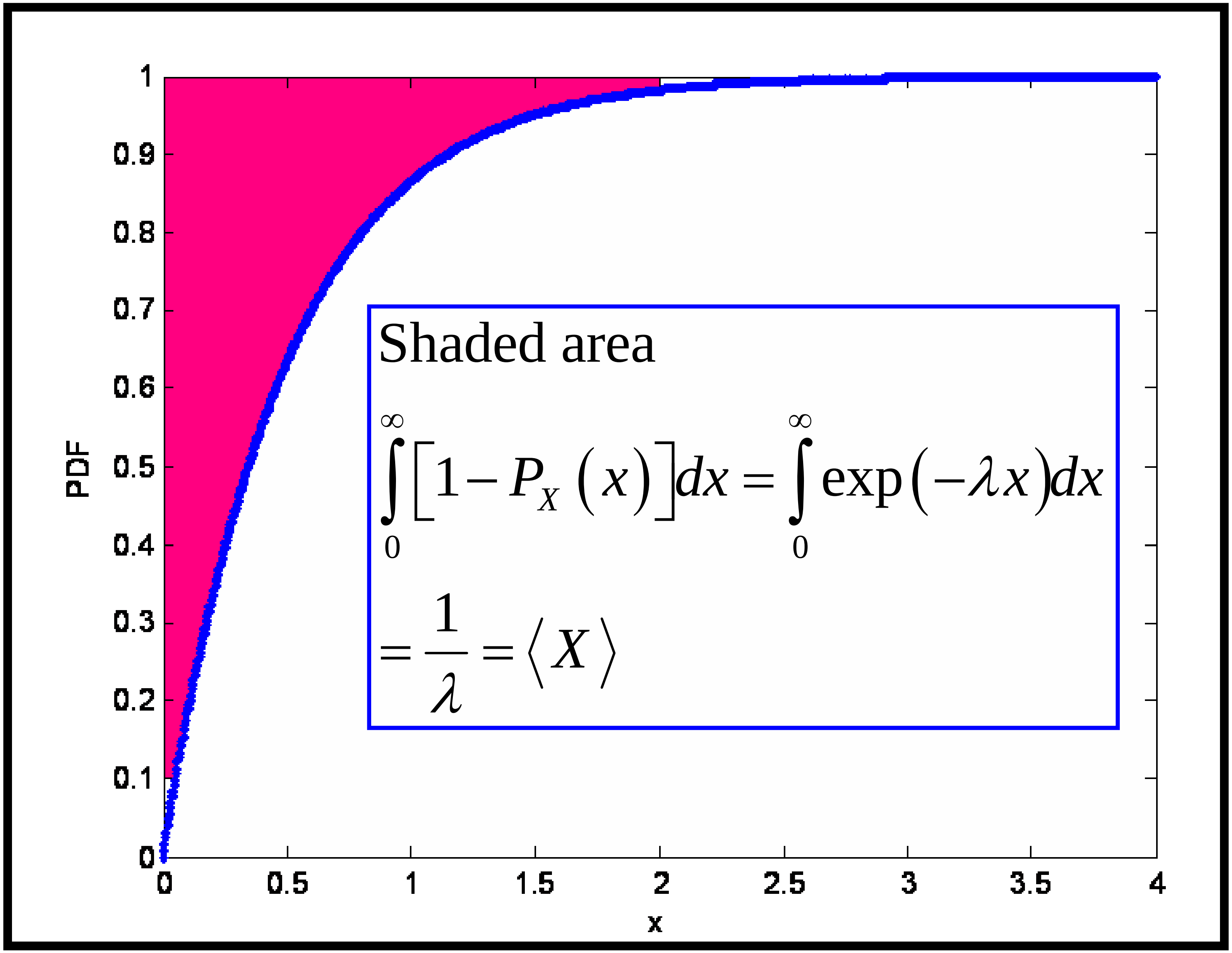
$$\text{Recall: } \left\langle X^n \right\rangle = \frac{1}{i^n} \frac{d^n \Phi_X(\omega)}{d\omega^n}\bigg|_{\omega=0}$$

$$\Phi_X(\omega) = \frac{\lambda}{(\lambda - i\omega)}$$

$$\frac{d\Phi_X(\omega)}{d\omega} = \frac{-\lambda(-i)}{(\lambda - i\omega)^2} \Rightarrow \frac{1}{i}\frac{d\Phi_X(\omega)}{d\omega}\bigg|_{\omega=0} = \frac{1}{\lambda} = \left\langle X \right\rangle$$

$$\frac{d^2\Phi_X(\omega)}{d\omega^2} = \frac{2\lambda(-i)^2}{(\lambda - i\omega)^3} \Rightarrow \frac{1}{i^2}\frac{d^2\Phi_X(\omega)}{d\omega^2}\bigg|_{\omega=0} = \frac{2}{\lambda^2} = \left\langle X^2 \right\rangle$$

$$\sigma_x^2 = \left\langle X^2 \right\rangle - \left\langle X \right\rangle^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$



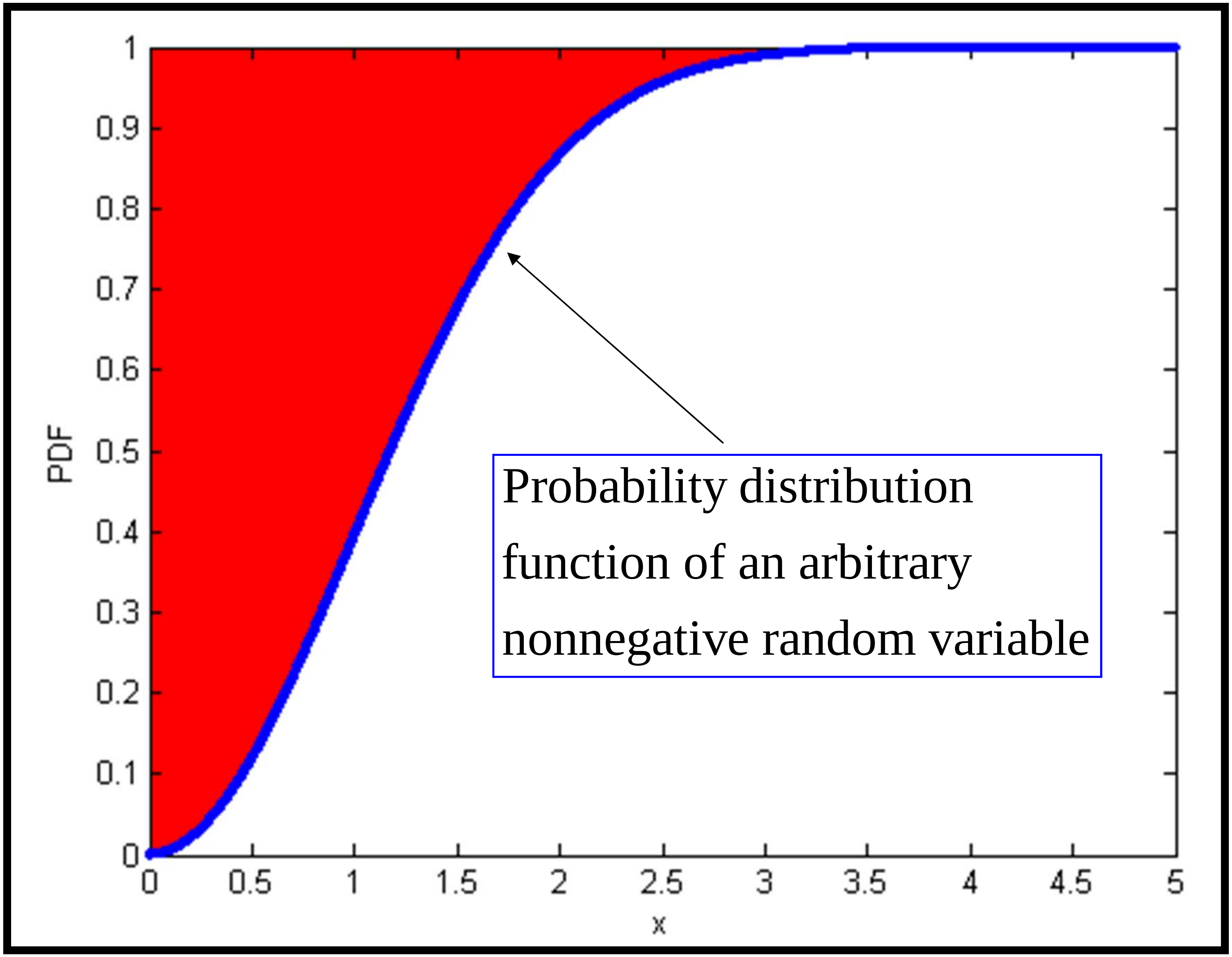
Shaded area
$$\int_0^\infty \left[1 - P_X(x)\right] dx = \int_0^\infty \exp(-\lambda x)\, dx$$
$$= \frac{1}{\lambda} = \langle X \rangle$$
PDF
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3
3.5
4

In fact this result is more generally true.

[Wentzel and Ovcharov, Applied problems in probability theory]

Let X be a non-negative random variable, that is, $P(X \le 0) = 0$. Show that the shaded area in the figure (next slide) is equal to the expected value of X; that is, show that

$$\mathrm{E}[X] = \int_0^\infty x p_X(x)\, dx = \int_0^\infty \left[1 - P_X(x)\right] dx.$$



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3
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4
4.5
5
x
Probability distribution function of an arbitrary nonnegative random variable

$$\mathrm{E}\left[X\right]=\int_{0}^{\infty} x p_X\left(x\right)dx$$

$$=\int_{0}^{\infty} x P_X'\left(x\right)dx=-\int_{0}^{\infty} x\left[1-P_X\left(x\right)\right]' dx$$

$$=-\left\{x\left[1-P_X\left(x\right)\right]\right\}_0^{\infty}+\int_{0}^{\infty}\left[1-P_X\left(x\right)\right]dx$$

$$\lim_{x\to 0} x\left[1-P_X\left(x\right)\right]\to 0 \quad \left[\text{Recall: } P_X\left(0\right)=0\right]$$

$$\lim_{x\to \infty} x\left[1-P_X\left(x\right)\right]\to 0 \ ?$$

$$\int_0^\infty x p_X(x)\,dx < \infty \Rightarrow \lim_{k\to\infty}\int_k^\infty x p_X(x)\,dx \to 0,$$

and, since $k\int_k^\infty p_X(x)\,dx \le \int_k^\infty x p_X(x)\,dx,$

$$\lim_{k\to\infty} k\left[1 - P_X(k)\right] \to 0$$

$$\Rightarrow \mathrm{E}[X] = \int_0^\infty \left[1 - P_X(x)\right]dx \text{ QED}$$

Problem 7

Let X be a random variable with pdf

$p_X(x) = \lambda \exp(-\lambda x); x \geq 0, \lambda = 2.$

Define $Z = \max(X, 2)$.

- Determine the pdf of Z. What kind of random variable is Z?
- Determine the characteristic function and hence evaluate the mean of Z.

$$Z = \max(X, 2) \Rightarrow$$

$$p_Z(z) = \lambda \exp(-\lambda z)\left[U(0) - U(2)\right] + P\left[X > 2\right]\delta(z-2)$$

Let $P\left[X > 2\right] = \exp(-2\lambda) = P_0$

Z is a mixed random variable.

$$\Phi_Z(\omega) = \int_0^2 \exp(i\omega z)\lambda \exp(-\lambda z)\,dz + \int_2^\infty \exp(i\omega z) P_0 \delta(z-2) dz$$

$$= P_0 \exp(2i\omega) + \lambda \left[\frac{\exp\left[-(\lambda - i\omega)z\right]}{-(\lambda - i\omega)} \right]_0^2$$

$$= P_0 \exp(2i\omega) + \frac{\lambda}{\lambda - i\omega}\left\{1 - \exp\left[-2(\lambda - i\omega)\right]\right\}$$

$$\Phi_Z(\omega) = P_0 \exp(2i\omega) + \frac{\lambda}{\lambda - i\omega}\left\{1 - \exp\left[-2(\lambda - i\omega)\right]\right\}$$

$$\frac{d\Phi_Z(\omega)}{d\omega} = P_0(2i)\exp(2i\omega) + \frac{-\lambda(-i)}{(\lambda - i\omega)^2}\left\{1 - \exp\left[-2(\lambda - i\omega)\right]\right\}$$

$$+\frac{\lambda}{\lambda - i\omega}\left\{-\exp\left[-2(\lambda - i\omega)\right](2i)\right\}$$

$$\left.\frac{d\Phi_Z(\omega)}{d\omega}\right|_{\omega=0} = 2P_0 i + \frac{i}{\lambda}\left\{1 - \exp(-2\lambda)\right\} - 2i\exp(-2\lambda)$$

$$\langle Z \rangle = 2P_0 + \frac{1}{\lambda}\left\{1 - \exp(-2\lambda)\right\} - 2\exp(-2\lambda)$$

Problem 8

Let X and Y be two independent standard normal random variables. Define $Z = \left|2X - 3Y\right|$.

Find the mean of Z.

$$\text{Define } U = 2X - 3Y \Rightarrow \langle U \rangle = 0$$

$$\langle U^2 \rangle = 4\langle X^2 \rangle + 9\langle Y^2 \rangle - 12\langle XY \rangle = 13$$

$$p_U(u) = \frac{1}{\sqrt{2\pi}\sqrt{13}} \exp\left(-\frac{1}{2}\frac{u^2}{13}\right); -\infty < u < \infty$$

$$p_U(u) = \frac{1}{\sqrt{2\pi}\sqrt{13}} \exp\left(-\frac{1}{2}\frac{u^2}{13}\right); -\infty < u < \infty$$

$$\sigma = \sqrt{13}$$

$$\langle |U| \rangle = \int_{-\infty}^{\infty} \frac{|u|}{\sqrt{2\pi}\sigma} \exp\left(-\frac{u^2}{2\sigma^2}\right) du = 2\int_{0}^{\infty} \frac{u}{\sqrt{2\pi}\sigma} \exp\left(-\frac{u^2}{2\sigma^2}\right) du$$

$$\text{Substitute } \frac{u^2}{2\sigma^2} = t \Rightarrow dt = \frac{u}{\sigma^2} du = dt$$

$$\Rightarrow \langle |U| \rangle = \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} \exp(-t)\, dt = \frac{2\sigma}{\sqrt{2\pi}} = \frac{2\sqrt{13}}{\sqrt{2\pi}} = 2.8768$$

Problem 9

Let X be a Cauchy random variable.

Prove that $Y = \dfrac{1}{X}$ is also a Cauchy random variable.

X is Cauchy $\Rightarrow$

$$p_X(x) = \frac{1/\pi}{1+x^2}; -\infty < x < \infty$$

$$Y = \frac{1}{X} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} = -y^2$$

$$p_Y(y) = \frac{p_X\left(\frac{1}{y}\right)}{y^2} = \frac{1/\pi}{1+\left(\frac{1}{y^2}\right)}\frac{1}{y^2} = \frac{1/\pi}{1+y^2} - \infty < y < \infty$$

Problem 10

Let X be a standard normal random variable.

Determine the pdf of

- $Y = X^2$
- $Y = \left| X \right|$
- $Y = \operatorname{sgn}\left(X \right)$
- $Y = \min\left(X, X^2 \right)$

$$X \sim N(0,1) \Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right); -\infty < x < \infty$$

- $Y = X^2 \Rightarrow X = \pm\sqrt{Y}$

$$\frac{dy}{dx} = 2x = \pm 2\sqrt{y}$$

$$p_Y(y) = \frac{p_X\left(\sqrt{y}\right) + p_X\left(-\sqrt{y}\right)}{2\sqrt{y}}$$

$$p_Y(y) = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2y^2}\right); 0 < y < \infty$$

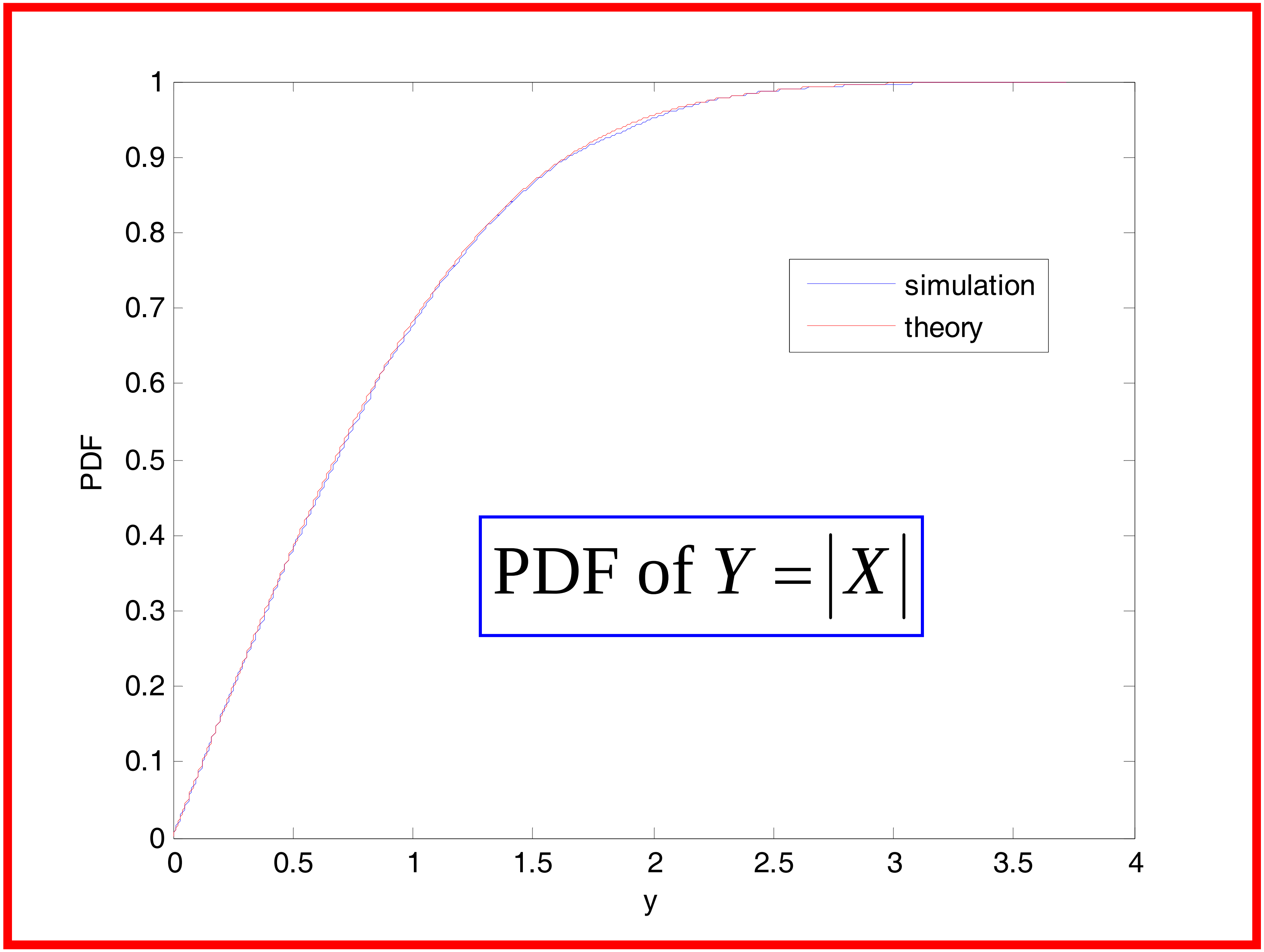
$$\bullet Y = |X|$$

$$P(Y \le y) = P(|X| \le y) = P(-y < X \le y)$$

$$= P(X \le y) - P(X \le -y)$$

$$= \Phi(y) - \Phi(-y); y \ge 0$$

$$p_Y(y) = 2\phi(y); y \ge 0$$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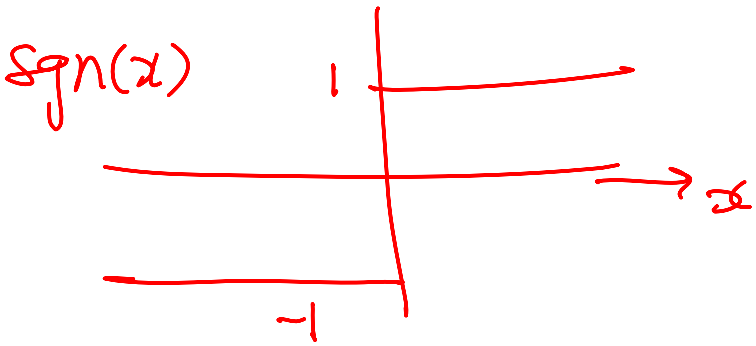
4
y
simulation
theory
PDF of $Y = |X|$

$\bullet Y = \mathrm{sgn}(X)$



$Y = 1 \text{ if } X > 0$

$= -1 \text{ if } X < 0$

$= 0 \text{ if } X = 0$

$$p_Y(y) = \delta(y-1)P(X>0) + \delta(y+1)P(X<0)$$

$$= \frac{1}{2}\left[\delta(y-1) + \delta(y+1)\right]$$

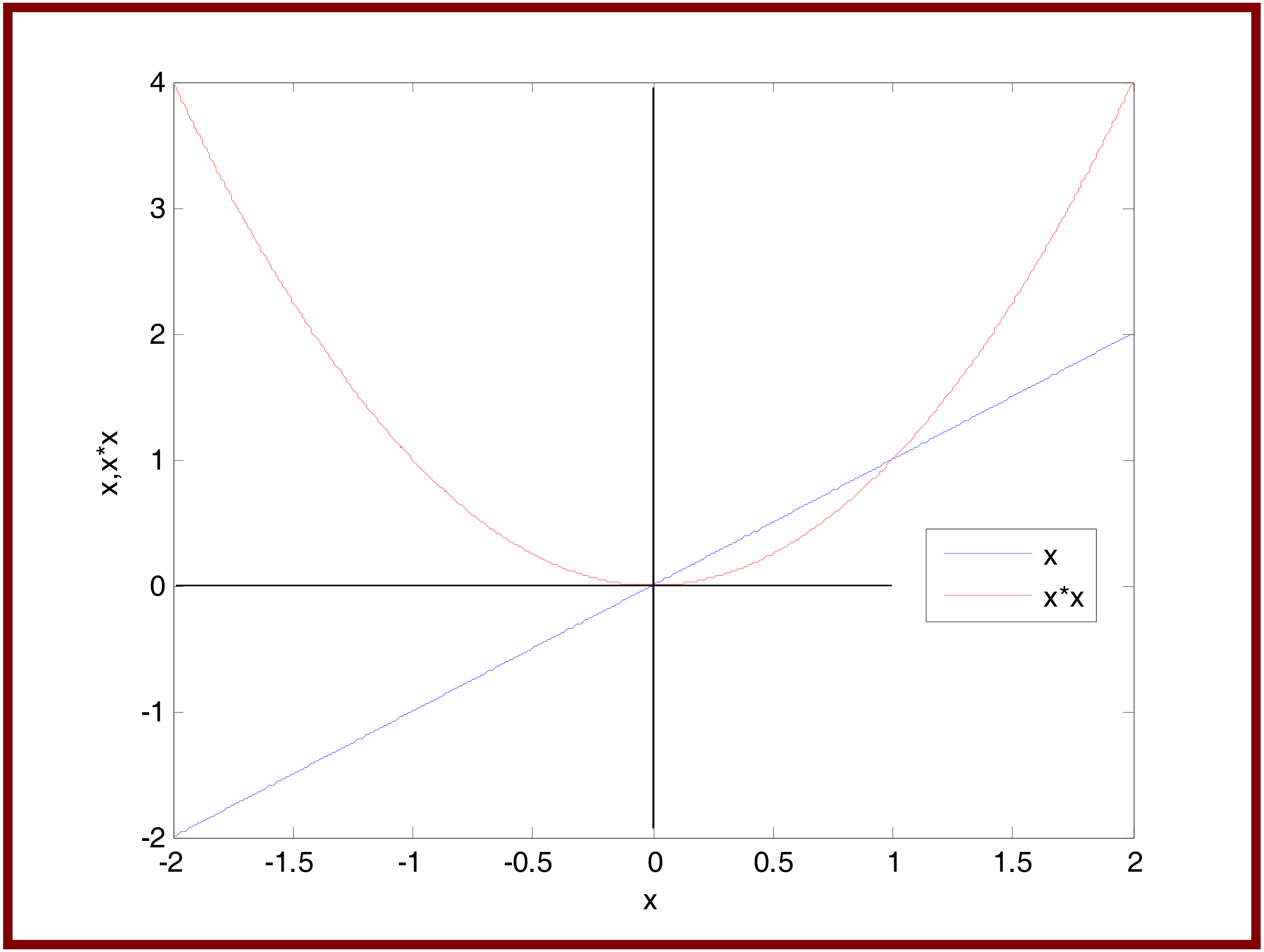
$$\bullet Y = \min\left(X, X^2\right)$$

$$P\left(Y \le y\right) = P\left(X^2 \le y\right) \text{ for } y \in \left(0,1\right)$$

$$= P\left(X \le y\right) \text{for } y \notin \left(0,1\right)$$

$$\Rightarrow p_Y\left(y\right) = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2y^2}\right) \text{for } y \in \left(0,1\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \text{for } y \notin \left(0,1\right)$$


4
3
2
1
0
-1
-2
x,x*x
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
x
x*x

Problem 11

Let X and Y be standard normal random variables.

Determine the pdf of $Z = |X| + |Y|$. $X \perp Y$

Define $U = |X|$ & $V = |Y|$

$$p_U(u) = 2\phi(u); u \geq 0$$

$$= 2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right); u \geq 0$$

$$p_V(v) = 2\phi(v); v \geq 0$$

$$= 2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right); v \geq 0$$

$$Z = U + V$$

$$p_Z(z) = \int_0^z p_U(u)\, p_V(z-u)\, du$$

$$= \int_0^z 2\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) 2\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-u)^2}{2}\right) du$$

$$= \frac{4}{2\pi} \int_0^z \exp\left(-\frac{1}{2}\left\{u^2 + (z-u)^2\right\}\right) du$$

$$= \frac{4}{2\pi} \int_0^z \exp\left(-\frac{1}{2}\left\{u + (z-u)\right\}^2 - \frac{1}{2}\left\{-2u(z-u)\right\}\right) du$$

$$= \frac{4}{2\pi} \exp\left(-\frac{z^2}{2}\right) \int_0^z \exp\left(-u^2 + uz)\right) du$$

$$= \frac{4}{2\pi} \exp\left(-\frac{z^2}{2}\right) \int_0^z \exp\left(-\left(u - \frac{z}{2}\right)^2 + \frac{z^2}{4}\right) du$$

$$p_Z(z) = \frac{4}{2\pi}\exp\left(-\frac{z^2}{2}\right)\int_0^z \exp\left(-\left(u-\frac{z}{2}\right)^2 + \frac{z^2}{4}\right)du$$

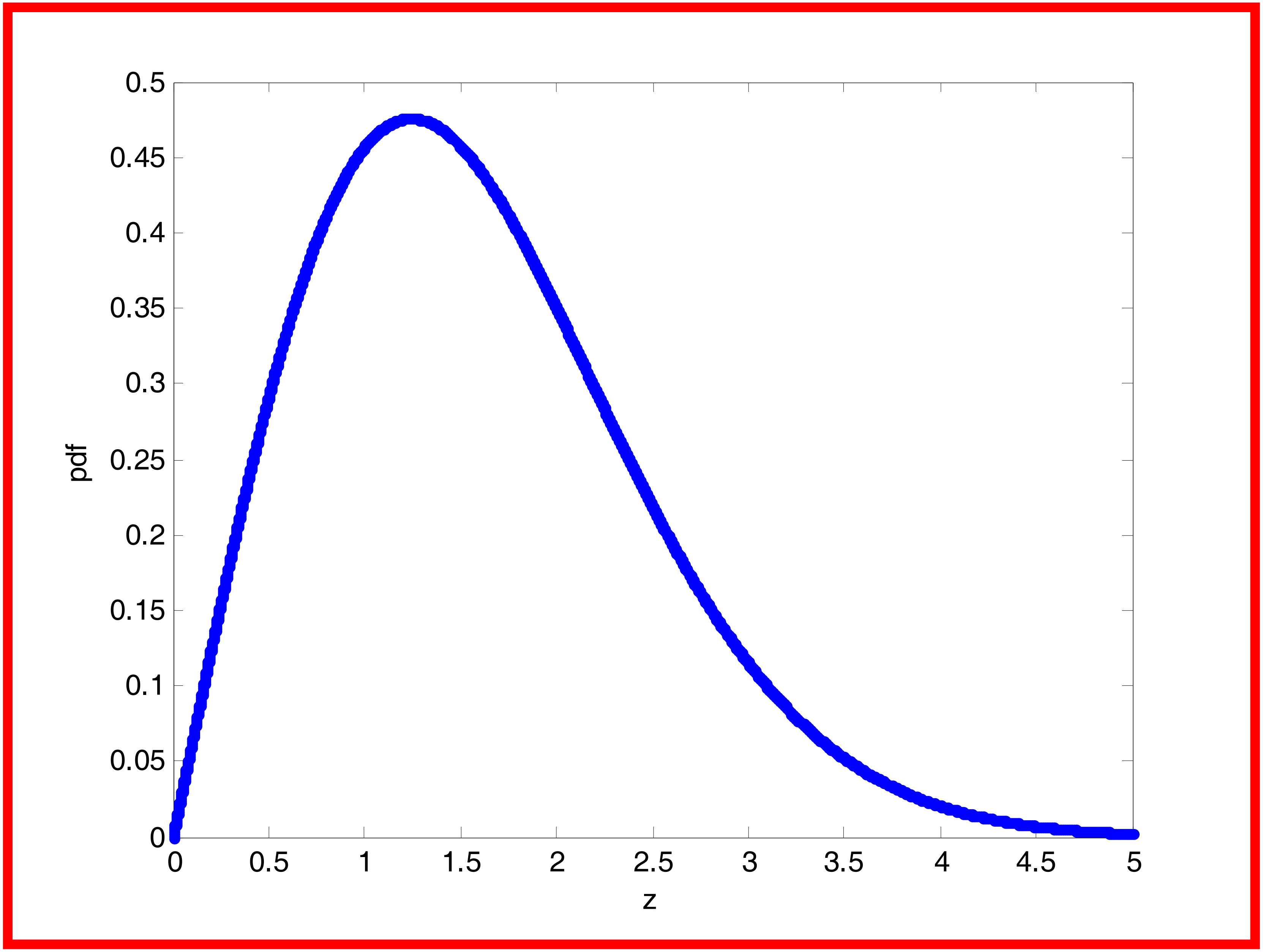
$$= \frac{4}{2\pi}\exp\left(-\frac{z^2}{4}\right)\int_0^z \exp\left(-\left(u-\frac{z}{2}\right)^2\right)du$$

$$\text{Put } u-\frac{z}{2} = \frac{t}{\sqrt{2}} \Rightarrow du = \frac{dt}{\sqrt{2}}$$

$$p_Z(z) = \frac{4}{2\pi}\frac{1}{\sqrt{2}}\exp\left(-\frac{z^2}{4}\right)\int_{-\frac{z}{\sqrt{2}}}^{\frac{z}{\sqrt{2}}} \exp\left(-\frac{t^2}{2}\right)dt$$

$$= \frac{8}{\sqrt{2\pi}}\frac{1}{\sqrt{2}}\exp\left(-\frac{z^2}{4}\right)\frac{1}{\sqrt{2\pi}}\int_0^{\frac{z}{\sqrt{2}}} \exp\left(-\frac{t^2}{2}\right)dt$$

$$\Rightarrow p_Z(z) = \frac{4}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right)\left\{\Phi\left(\frac{z}{\sqrt{2}}\right)-\frac{1}{2}\right\}$$


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
pdf
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
z

Problem 12

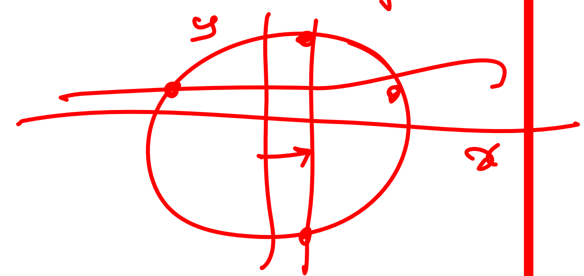
Consider two random variables X and Y with

$$p_{XY}(x,y)=\begin{cases} C\forall x,y \ni \sqrt{x^2+y^2}\le 2. \\ 0 \text{ otherwise} \end{cases}$$

- Find C
- Find marginal pdf-s of X and Y and verify if X and Y are independent.
- Select a point B inside the circular region $\sqrt{x^2+y^2}\le 2$ and let (R,Θ) be the polar cooridnates of B. Determine the joint pdf of R and Θ, mariginal pdf-s of R and Θ. Verify if R and Θ are independent.

$$p_{XY}(x,y)=\begin{cases} C \ \forall x,y \ni \sqrt{x^2+y^2}\le 2. \\ 0 \text{ otherwise} \end{cases}$$

$$\Rightarrow \iint\limits_{\sqrt{x^2+y^2}\le 2} C\,dx\,dy=\int_0^{2\pi}\int_0^{2} Cr\,dr\,d\theta = 4\pi C = 1$$



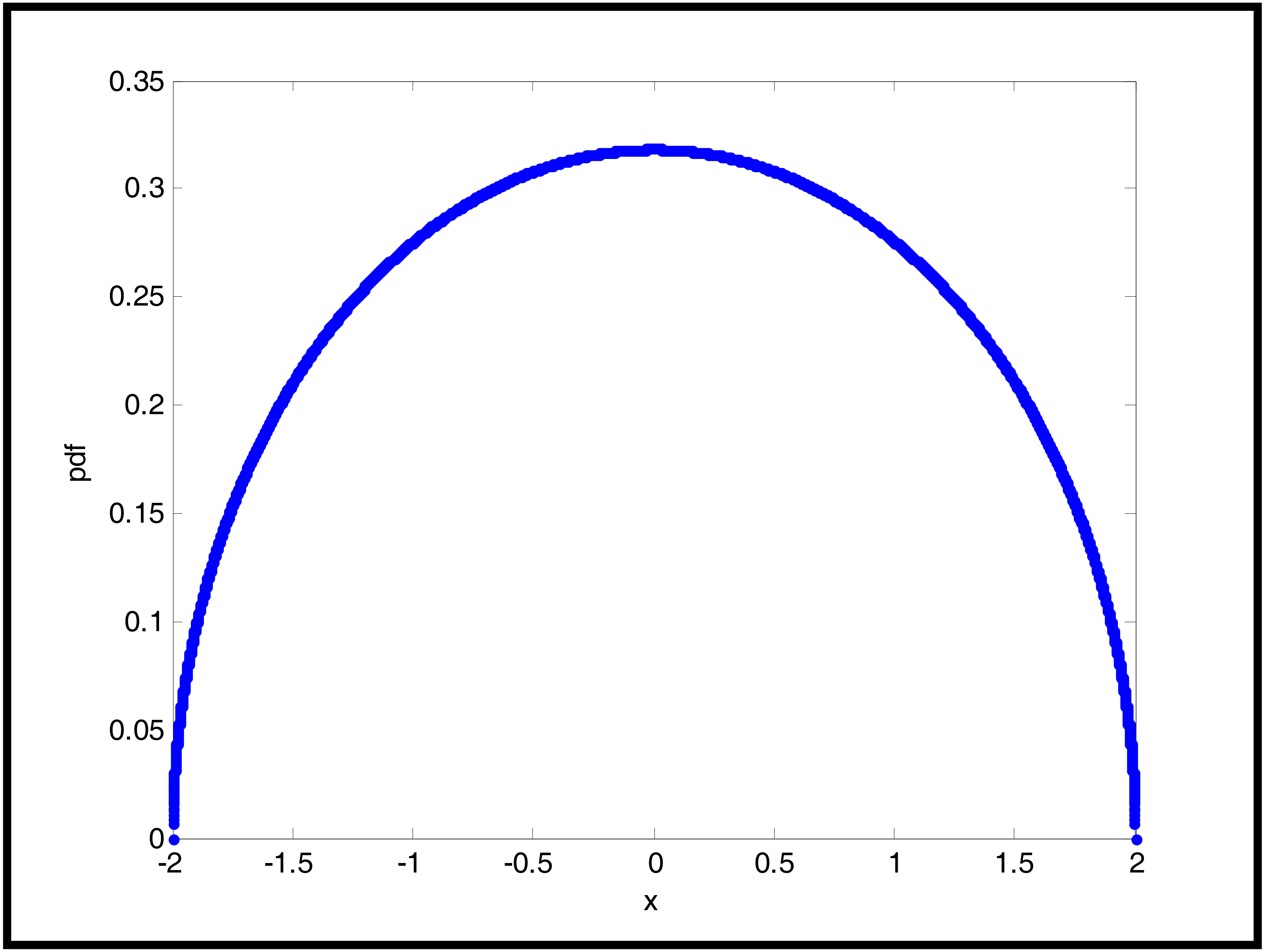
$$\Rightarrow C=\frac{1}{4\pi}$$

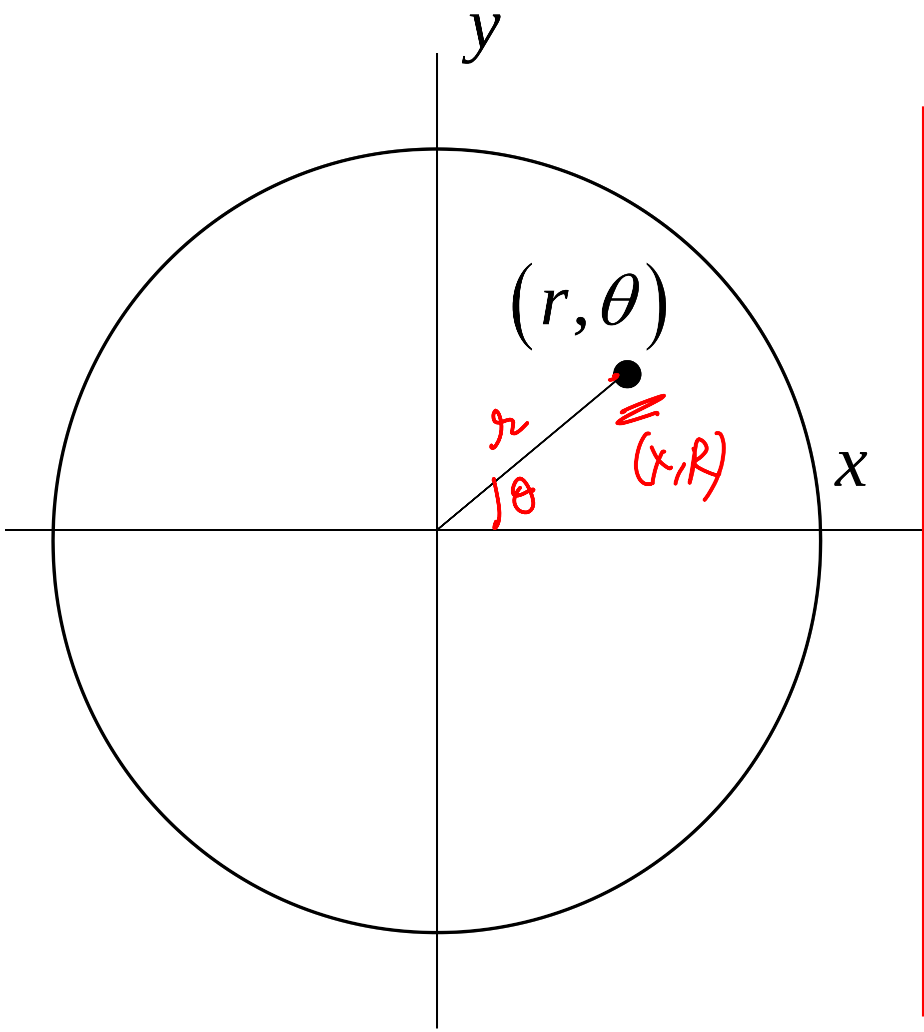
$$p_X(x)=\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\frac{1}{4\pi}dy=\frac{1}{2\pi}\sqrt{4-x^2};-2<x<2$$

$$p_Y(y)=\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\frac{1}{4\pi}dx=\frac{1}{2\pi}\sqrt{4-y^2};-2<y<2$$

$$\Rightarrow p_{XY}(x,y)\ne p_X(x)\,p_Y(y)$$

$\Rightarrow X$ and Y are not independent.


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
pdf
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x



$$X = R\cos\Theta$$

$$Y = R\sin\Theta$$

$$p_{R\Theta}(r,\theta) = \frac{r}{4\pi}; 0 < r < 2; 0 < \theta < 2\pi$$

$$p_R(r) = \frac{r}{2}; 0 < r < 2$$

$$p_\Theta(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

$$p_{R\Theta}(r,\theta) = p_R(r)\, p_\Theta(\theta)$$

R and Θ are independent.

Problem 13

Let $\left(X_i\right)_{i=1}^{n}$ be an iid sequence with the common pdf given by

$$p_X\left(x\right)=\lambda\exp\left(-\lambda x\right); x\geq 0.$$

Define $Y=\sum_{i=1}^{n}X_i$. Determine the pdf of Y for $\lambda=2$ and n=10.

$x_1 + x_2$

$y_1 + x_3$

$y_2 + x_4$ y_3 . . . .

Interpret exponetial distribution as a model for inter-arrival time between Poisson points.

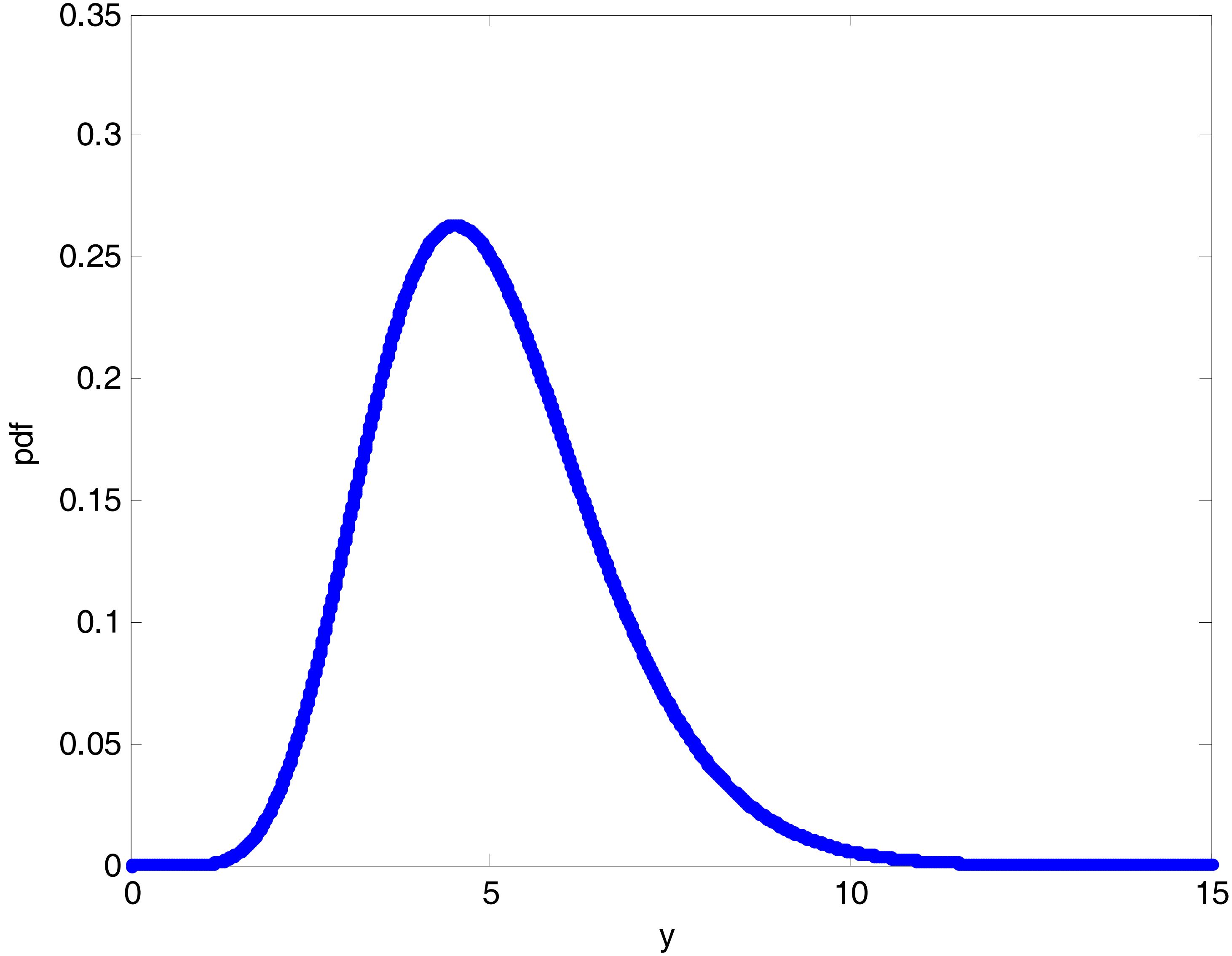
$$p_Y\left(y\right)dy = P\left[y < Y \le y + dy\right]$$

$$= P\begin{bmatrix}\text{exactly } n\text{-1 points occur in 0 to } y \text{ and} \\ \text{one event in } y \text{ to } y+dy\end{bmatrix}$$

$$= \frac{\left(\lambda y\right)^{n-1}}{\left(n-1\right)!}\lambda \exp\left(-\lambda y\right)dy$$

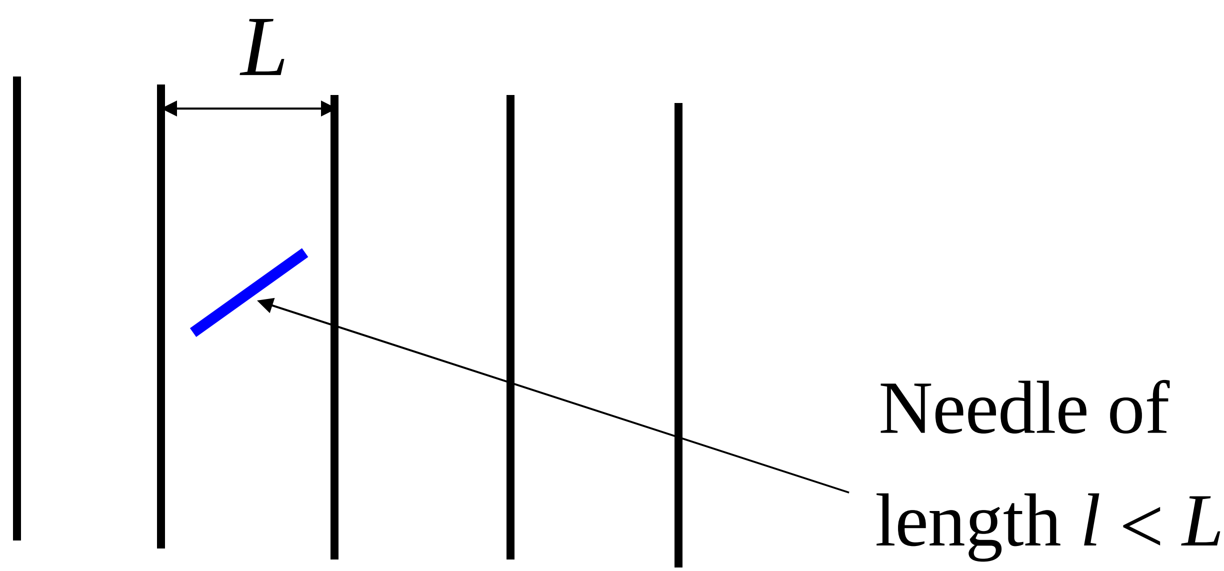
$$\Rightarrow p_Y\left(y\right) = \frac{\lambda\left(\lambda y\right)^{n-1}}{\left(n-1\right)!}\exp\left(-\lambda y\right); y \ge 0$$

Erlang pdf

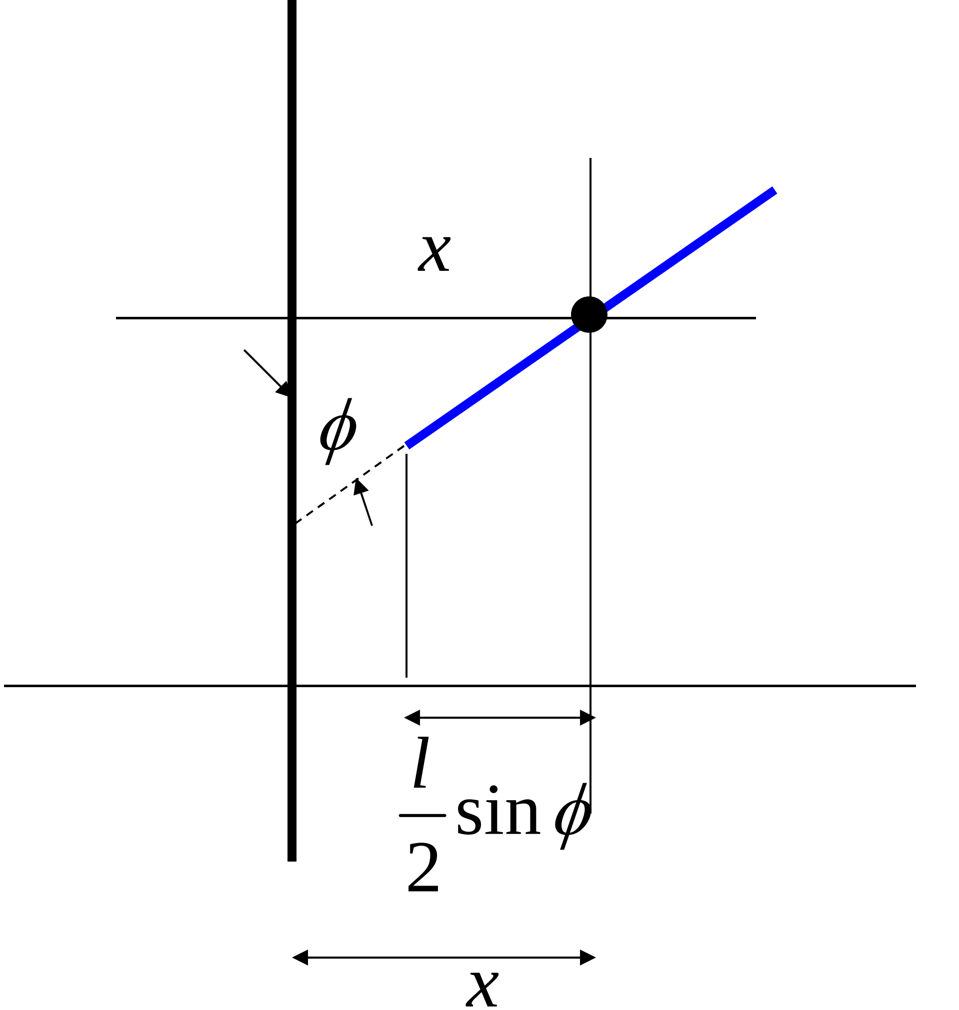

0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
pdf
0
5
10
15
y

Problem-14 [Buffon's needle problem]

A set of n parallel lines equidistant from each other is drawn on a plane. The lines are at a distance of L from each other. A needle of length l is placed randomly on the plane. Find the probability that the needle would intersect one of the lines.



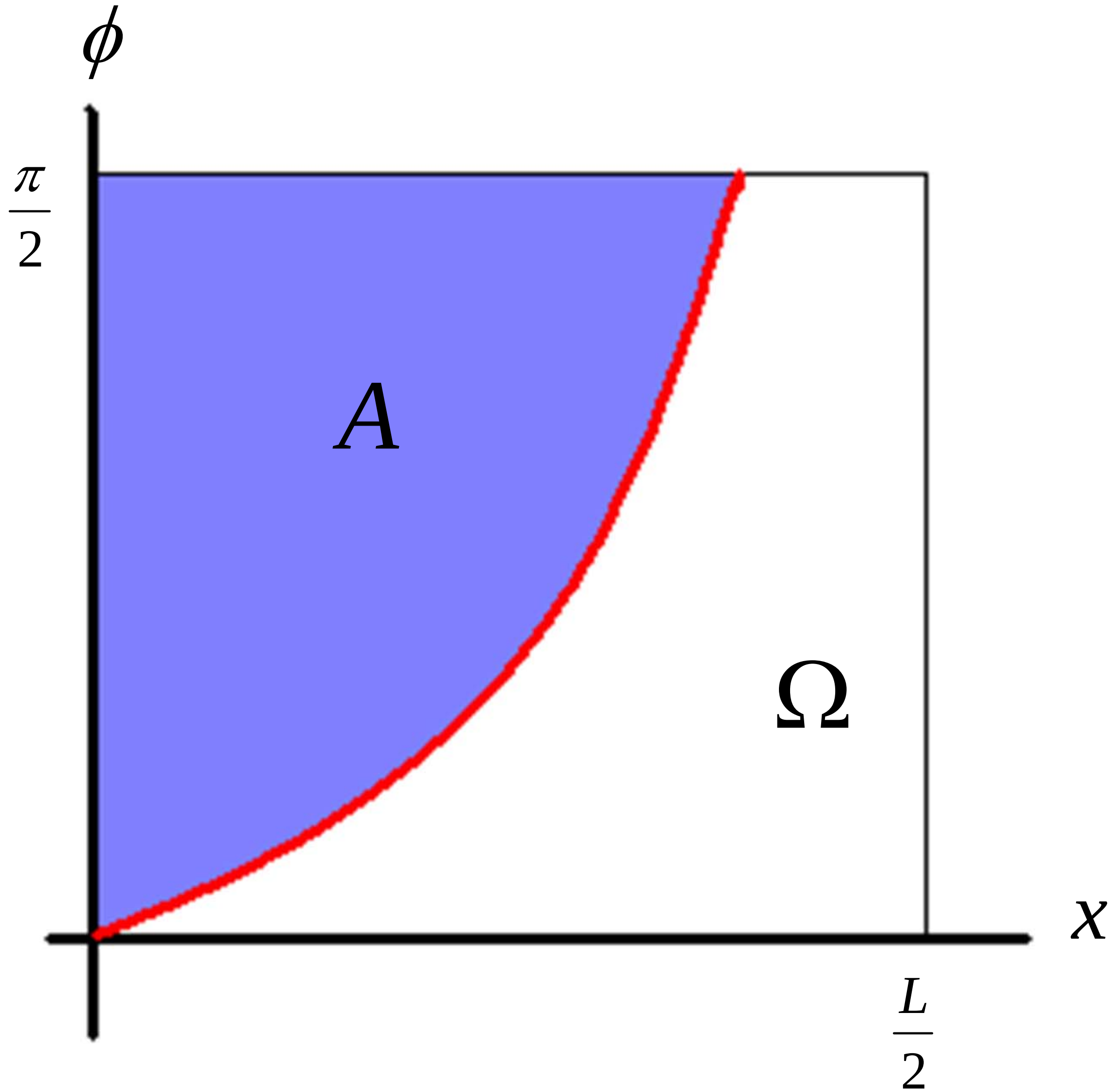
$x =$ distance from the midpoint of the needle to the nearest line the left



$$x \sim U\left(0, \frac{L}{2}\right)$$

$$\phi \sim U\left(0, \frac{\pi}{2}\right)$$

$$x \perp \phi$$



ϕ
$\frac{\pi}{2}$
A
Ω
x
$\frac{L}{2}$

$$P\left(\text{needle intersects the line}\right) = P\left(x \le \sin\phi\right)$$

$$P\left(x \le \sin\phi\right) = \frac{\text{Shaded area}}{\text{Total area}}$$

$$= \frac{\int_0^{\frac{\pi}{2}} \frac{l}{2}\sin\varphi\, d\varphi}{\frac{\pi L}{4}} = \frac{2l}{\pi L}$$

Remark: this result can be used to estimate value of π

Problem 15

- Let X be a normal random variable with mean m and standard deviation σ. Show that $\langle X^n \rangle = m\langle X^{n-1} \rangle + (n-1)\sigma^2 \langle X^{n-2} \rangle$ $n = 3, 4, \cdots$
- Let X and Y be two normal random variables such that

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} \sim N\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\right]$$

Define

$$U = 2 + 4X + 10XY$$

$$V = 1 + 2XY + 6Y^2$$

  - Find mean and covariance matrix of U and V.

Hint

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} \sim N\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\right]$$

X and Y are uncorrelated and Gaussian

$\Rightarrow X$ and Y are independent

Introduce

$$X_1 = \frac{X-1}{2} \Rightarrow \langle X_1 \rangle = 0, \sigma_{X_1}^2 = 1$$

$$X_2 = \frac{Y-2}{3} \Rightarrow \langle X_2 \rangle = 0, \sigma_{X_2}^2 = 1$$

$$U = 2 + 4(2X_1 + 1) + 10(2X_1 + 1)(3X_2 + 2)$$

$$= 26 + 48X_1 + 30X_2 + 60X_1X_2$$

$$V = 1 + 2XY + 6Y^2$$

$$= 1 + 2(2X_1 + 1)(3X_2 + 2) + 6(3X_2 + 2)^2$$

$$= 29 + 8X_1 + 78X_2 + 12X_1X_2 + 54X_2^2$$

Hint

$$U = 26 + 48X_1 + 30X_2 + 60X_1X_2$$

$$V = 29 + 8X_1 + 78X_2 + 12X_1X_2 + 54X_2^2$$

$$\langle U \rangle = \langle 26 + 48X_1 + 30X_2 + 60X_1X_2 \rangle = 26$$

$$\langle V \rangle = \langle 29 + 8X_1 + 78X_2 + 12X_1X_2 + 54X_2^2 \rangle = 29 + 54 = 83$$

$$U_1 = U - 26 = 48X_1 + 30X_2 + 60X_1X_2$$

$$V_1 = V - 83 = 8X_1 + 78X_2 + 12X_1X_2 + 54X_2^2 - 54$$

$$\sigma_U^2 = \langle U_1^2 \rangle = \left\langle \left(48X_1 + 30X_2 + 60X_1X_2\right)^2 \right\rangle$$

$$\sigma_U^2 = \langle V_1^2 \rangle = \left\langle \left(8X_1 + 78X_2 + 12X_1X_2 + 54X_2^2 - 54\right)^2 \right\rangle$$

$$\sigma_{UV} = \langle U_1 V_1 \rangle = < \left(48X_1 + 30X_2 + 60X_1X_2\right)$$

$$\left(8X_1 + 78X_2 + 12X_1X_2 + 54X_2^2 - 54\right) >$$

Use $\langle X^n \rangle = m \langle X^{n-1} \rangle + (n-1)\sigma^2 \langle X^{n-2} \rangle$

Problem 16

Let $X_1, X_2, \cdots, X_n$ be an iid sequence of random variables with common PDF $P_X(x)$.

Let $X_1, X_2, \cdots, X_n$ be arranged in an increasing order

$X_{1:n} \le X_{2:n} \le X_{3:n} \le \cdots \le X_{n:n}$.

Obtain the PDF of $X_{r:n}$ where $1 \le r \le n$.

Remark

- Notice

$$X_{1:n} = \min(X_1, X_2, \cdots, X_n)$$

$$X_{n:n} = \max(X_1, X_2, \cdots, X_n)$$

- We have already obtained the PDF of $X_{1:n}$ & $X_{n:n}$.

Consider a trial in which a sample of $X_1, X_2, \cdots, X_n$ is observed.

Define $(\text{Success}) = \{X_j \le x\}$ for a specified value of x.

Define $M_n(x) =$ Number of elements in the sample for with values $X_j \le x$.

$M_n(x)$ is a random variable following Binomial distribution with $p = \{X_j \le x\} = P_X(x) \Rightarrow$

$$P[M_n(x) = k] = {}^nC_k\, p^k (1-p)^{n-k}\,; k = 0, 1, 2, \cdots, n$$

$$\Rightarrow P[M_n(x) \le r] = \sum_{k=0}^{r} {}^nC_k [P_X(x)]^k [1 - P_X(x)]^{n-k}$$

$$P\left[M_n(x)\le r\right]=\sum_{k=0}^{r}{}^{n}C_k\left[P_X(x)\right]^k\left[1-P_X(x)\right]^{n-k}$$

Conisder the event

$$\left\{X_{r:n}\le x\right\}=\left\{r \text{ or more elements have values greater than } x\right\}$$

$$=\left\{M_n(x)\ge r\right\}$$

$$\Rightarrow$$

$$P\left\{X_{r:n}\le x\right\}=P\left\{M_n(x)\ge r\right\}=1-P\left\{M_n(x)\le r-1\right\}$$

$$=1-\sum_{k=0}^{r-1}{}^{n}C_k\left[P_X(x)\right]^k\left[1-P_X(x)\right]^{n-k}$$

$$P_{X_{r:n}}(x)=\sum_{k=r}^{n}{}^{n}C_k\left[P_X(x)\right]^k\left[1-P_X(x)\right]^{n-k}$$