NPTEL Course on Stochastic Structural Dynamics

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Assignment

Probability and random variables

1. Let A and B be two events. Show that $A \subset B \Rightarrow \mathrm{P}(A) \le \mathrm{P}(B)$.
2. Two random variables may have the same mean and variance but they need not have the same pdf. Justify this statement.
3. X is a discrete random variable with $\mathrm{P}[X=1]=1,\ \mathrm{P}[X=-1]=q\ \&\ \mathrm{P}[X=0]=1-p-q$. Find the characteristic function of X and hence determine its mean and variance.
4. X and Y are discrete iid random variables with PDF as in the previous example. Determine $\mathrm{P}[X=Y]\,\&\,\mathrm{P}[X>Y]$.
5. $\{X_i\}_{i=1}^{n}$ is a sequence of discrete iid random variables with PDF as in problem 3. Define $S_n=\sum_{i=1}^{n} X_i$. Find the mean and standard deviation of S_n. Evaluate $\mathrm{P}[S_n=0]$.
6. X is an exponential random variable with pdf given by $p_X(x)=\alpha\exp(-\alpha x); x\ge 0$. Obtain the moment generating and characteristic functions for X. Evaluate the probability that X exceeds mean plus twice the standard deviation.
7. X and Y are two Gaussian random variables with the jpdf given by

$$p_{XY}(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{\left(1-r_{XY}^2\right)}}\exp\left[-\frac{1}{2\left(1-r_{XY}^2\right)}\left\{\left(\frac{x-\eta_X}{\sigma_X}\right)^2+\left(\frac{y-\eta_Y}{\sigma_Y}\right)^2-\frac{2r_{XY}(x-\eta_X)(y-\eta_Y)}{\sigma_X\sigma_Y}\right\}\right]$$

$-\infty<x<\infty; -\infty<y<\infty$

Show that

- The above function is a valid pdf.
- $\langle X\rangle=\eta_X;\mathrm{Var}(X)=\sigma_X^2;\mathrm{Covariance}(X,Y)=r_{XY}\sigma_X\sigma_Y$
- $p_Y(y\,|\,X=x)$ is Gaussian and hence find the conditional mean and variance.

8. X and Y are as in the preceding problem. It is given that $\eta_X=1;\eta_Y=2;\sigma_X=2;\sigma_Y=4;r_{XY}=0.3$. Introduce two independent and identical distributed random variables U and V such that they have the common pdf which is a normal pdf with zero mean and unit standard deviation. A transformation of the form

$$\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}+\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}\begin{pmatrix}U\\V\end{pmatrix}$$

is given to exist. Determine $a, b, t_{ij}; i, j = 1, 2$.

9. X is a normal random variable with zero mean and unit standard deviation. Define $Y = X^3$. Find correlation coefficient between X and Y. Repeat the exercise for $Y = X^4$. Comment on the results obtained.
10. A simply supported beam carries two concentrated loads P and Q at quarter points. The two loads are jointly normal with $\begin{Bmatrix} P \\ Q \end{Bmatrix} \sim N\left[\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}\right]$. Determine the pdf of the midspan displacement and the joint pdf of the support reactions.

11. Let X be the stress and Y be the displacement at a specified location in a randomly loaded structure. Write down the units (in SI system) for the following quantities:
    a. Covariance of X and Y
    b. JPDF of X and Y
    c. Conditional PDF of X given $Y=y$.
    d. n-th order moment of Y.
    e. Joint characteristic function of X and Y
    f. Joint pdf of X and Y
12. X and Y are two random variables having jpdf given by

$$p_{XY}(x, y) = N \exp\left[\left(x^2 + y^2\right) - \left(x^2 + y^2\right)^2\right]; -\infty < x, y < \infty$$

    Examine if X and Y are (i) uncorrelated, (ii) independent.
13. Let X be a normal random variable with mean m and standard deviation σ. Define

$$Y = \begin{cases} 2X & \text{if } |X| \le 2\sigma \\ -2X & \text{if } |X| > 2\sigma \end{cases}$$

    Determine the pdf of Y.
14. U and V are iid random variables distributed uniformly in 0 to 1. Determine the pdf of the random variables (a) $2U+3V$, (b) UV, (c) U/V, (d) max(U,V), and (e) min(U,V).
15. X and Y are two continuous random variables. X has a normal distribution with zero mean and unit standard deviation. For a given value of $X=x$, Y is also normal distributed with mean x and unit standard deviation. Find jpdf of X and Y, pdf of Y, and conditional pdf of Y given $X=x$.

<u>Random processes</u>

16. Show that a second order strong sense stationarity of a random process implies
    a. 1st order strong sense stationarity, and

b. 2nd order wide sense stationarity.

17. *A,B*, and *C* are three jointly Gaussian random variables with

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} \sim N\left[\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0.4 \\ 0 & 1 & 0 \\ 0.4 & 0 & 1 \end{bmatrix}\right]$$

Define two random processes

$$X(t) = A + 2Bt^2$$
$$Y(t) = 0.5A + Ct + Bt^2$$

Find the autocovariance of processes *X(t)* and *Y(t)*. Write down the joint pdf of *X(1)* and *Y(2)*.

18. Let $\{X_i(t)\}_{i=1}^n$ be a set of stationary, zero mean, Gaussian random processes and $\{a_i, \tau_i\}_{i=1}^n$ be a set of deterministic constants. Define $Y(t) = \sum_{i=1}^n a_i X(t - \tau_i)$. Determine the autocovariance and PSD functions of the process *Y(t)*.
19. Let *Y(x)* be a random process representing the road roughness measured in mm. Write down the units of PSD, autocovariance, 1^{st} order PDF, 1^{st} order pdf, and *n*-th order pdf of *Y(x)*.
20. Let *X(t)* be a zero mean stationary Gaussian random process. Justify the statement that *X(t)* is completely specified in terms of its PSD function. Also, show that $\langle X(t)\dot{X}(t)\rangle = 0$.
21. Consider a random process defined by $X(t) = a\cos(\lambda t + \phi)$. Here a and λ are deterministic and ϕ is a random variable distributed uniformly in 0 to 2π. Examine if the process is ergodic in mean and ergodic in autocorrelation.
22. *X(t)* is a zero mean stationary random process. Define $Y(t) = X(t-\alpha) + S(t)$ where α is a deterministic constant and $S(t)$ is a zero mean stationary white noise process that is independent of *X(t)*. Determine the matrix of the autocovariance functions and power spectral density functions for the processes *X(t)* and *Y(t)*.
23. *X(t)* is a stationary Gaussian random process with zero mean and autocovariance function given by $R_{XX}(\tau) = 2\exp(-\alpha\tau^2)\cos\beta\tau$. Show that the mathematical PSD function is given by

$$S_{XX}(\omega) = \sqrt{\frac{\pi}{\alpha}}\left\{\exp\left(-\frac{(\omega-\beta)^2}{4\alpha}\right) + \exp\left(-\frac{(\omega+\beta)^2}{4\alpha}\right)\right\}$$

Evaluate the autocovariance and PSD function of the derivative of *X(t)*.

24. For the random process described in the preceding problem, obtain the jpdf of the envelope and phase process.
25. Let *X(t)* and *Y(t)* be two zero mean random processes. Show that

$$\left\langle \left.\frac{d^n X}{dt^n}\right|_{t=t_1} \left.\frac{d^m X}{dt^m}\right|_{t=t_2} \right\rangle = \frac{\partial^{n+m} R_{XX}\left(t_1,t_2\right)}{\partial t_1^n \partial t_2^m}$$

Furthermore, if *X(t)* and *Y(t)* are jointly stationary show that

$$\left\langle \frac{d^n X\left(t+\tau\right)}{dt^n} \frac{d^m Y\left(t\right)}{dt^m} \right\rangle = \left(-1\right)^m \frac{d^{n+m} R_{XY}\left(\tau\right)}{d\tau^{n+m}}$$

26. Let *A,B,C*, and *D* be jointly Gaussian random variables. Investigate the conditions under which the vector random process

$$\begin{Bmatrix} X\left(t\right) \\ Y\left(t\right) \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} 1 \\ t \end{Bmatrix}$$

is Markov in nature.

<u>Random vibration analysis</u>

27. A first order system excited by a harmonic excitation and a Gaussian white noise process with unit strength can be described by

$$\begin{aligned} &\dot{x} + \alpha x = P\cos\lambda t + w(t) \\ &x(0) = x_0 \\ &< w(t) >= 0 \\ &< w(t_1)w(t_2) >= I\delta(t_1 - t_2) \end{aligned}$$

Here $\alpha > 0$ is a deterministic constant. Set up the expression for the mean and autocovariance of the response. Discuss the nature of response as $t \to \infty$.

28. The steady state response variance of a sdof system under Gaussian white noise with intensity *I* has been shown to be given by

$$\sigma_{xs}^2 = \int_0^\infty \frac{(I/\pi)}{m^2\left[\left(\omega^2 - \Omega^2\right)^2 + \left(2\eta\omega\Omega\right)^2\right]} d\Omega$$

Evaluate this integral using the residue theorem.

29. A sdof system driven by a non-Gaussian random excitation is governed by

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f^2(t); x(0) = 0; \dot{x}(0) = 0$$
$$\dot{f} + \beta f = w(t); f(0) = 0$$

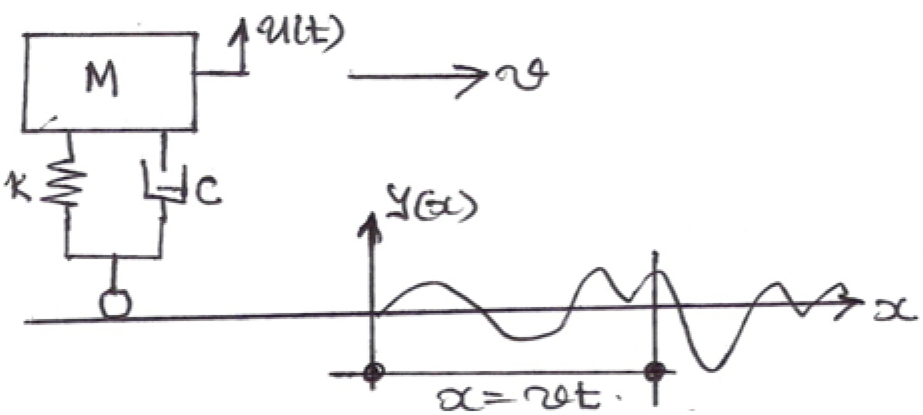
where *w(t)* is a zero mean white noise process with $\langle w(t) w(t+\tau) \rangle = \delta(\tau)$. Set up the governing FPK equation for the evolution of the tpdf. Also, setup the governing equations for the time evolution of first two order moments and evaluate the steady state response. Is the closure problem encountered here? Analyze the problem by using the Duhamel integra/FRF based approach and compare the results with those obtained using the FPK equation approach.

30. A randomly driven sdof system is governed by the equation

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = \dot{f} + \alpha f$$

Here *f(t)* is given to be a zero mean stationary random process. Determine the PSD matrix of the displacement, velocity and acceleration response process in the steady state.

31. A vehicle running on a rough road with constant velocity is idealized as a sdof system as shown in the figure below.



Vehicle running on a rough road.

To a first approximation the road profile is modeled as $y(x) = A\cos\lambda x + B\sin\lambda x$ where A and B are zero mean Gaussian iid random variables with variance σ^2 and λ is a deterministic constant. Determine the steady state variance of the force transmitted to the mass. Discuss how you would select the spring and the dashpot so that this variance is minimized.

32. In the study of vortex induced response of long span bridges and tall towers it is of interest to analyze the stationary response of system governed by

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x=P\cos\left(\lambda t+\phi\right)+S\left(t\right);x(0)=x_0,\dot{x}\left(0\right)=\dot{x}_0$$

Here ϕ is a random variable that is uniformly distributed in $\left[-\pi,\pi\right]$, *S(t)* is a zero mean stationary Gaussian random process which is independent of ϕ, and *P* is a deterministic constant. Does the system reach a stochastic steady state? If so, determine the response autocovariance and PSD function in the steady state. Also, determine the jpdf of displacement and velocity processes in the steady state. Are these processes, at the same time instant in the steady state, uncorrelated? Independent?

33. The equation of motion of a sdof system with structural damping is given by

$$m\ddot{x}+k\left[1+ig\right]x=F(t)$$

Here *F(t)* is a zero mean, Gaussian white noise process with intensity *I* and $i=\sqrt{-1}$. Determine the PSD of the response in the steady state and hence determine the variance of the displacement and velocity processes.

34. An undamped sdof system is set into free vibration by imparting an initial displacement x_0 and velocity $\dot{x}_0$. The initial conditions are given to be random with x_0 & $\dot{x}_0$ being jointly Gaussian with the mean vector being identically equal to zero. Determine the conditions under which the response process can become stationary and for this case determine the average rate of crossing of level $\pm\alpha$.

35. With reference to a discrete mdof system, we define $h_{ij}\left(t\right)$ and $H_{ij}\left(\omega\right)$ to be respectively the impulse response and frequency response functions for drive station *i* and measurement station *j*. Show that

$$H_{ij}\left(\omega\right)=\int_{-\infty}^{\infty}h_{ij}\left(t\right)\exp\left(i\omega t\right)d\tau$$

$$h_{ij}\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H_{ij}\left(\omega\right)\exp\left(-i\omega t\right)d\omega$$

36. Let $g\left(x,\xi,t\right)$ & $G\left(x,\xi,\omega\right)$ denote respectively the Green's function for a beam in time and frequency domains. Show that $g\left(x,\xi,t\right)$ & $G\left(x,\xi,\omega\right)$ form a Fourier transform pair, that is, show that,

$$G(x,\xi,\omega)=\int_{-\infty}^{\infty} g(x,\xi,t)\exp(i\omega t)\,dt$$

$$g(x,\xi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(x,\xi,\omega)\exp(-i\omega t)\,d\omega$$

37. *X(t)* is a nonstationary Gaussian random process with time varying mean and variance. Determine the average fractional occupation time above the levels $\pm\alpha(t)$.

38. Consider two adjoining buildings which are separated by a distance *L*. The question of possible pounding between the two buildings during the event of an earthquake needs to be analyzed. As a first approximation, the two buildings are modeled as sdof systems and the earthquake ground acceleration is modeled as a zero mean stationary Gaussian random process. Let us limit our attention to the building response in the steady state. Determine the average number of times the two buildings would pound against each other and hence determine the time for the first occurrence of pounding. It may be assumed that the occurrence of pounding is a rare event.

39. A lightly damped sdof system driven by a zero mean, stationary, Gaussian, bandlimited white noise excitation is governed by the equation

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x=f(t)$$

The system may be taken to start from rest. Using the method of stochastic averaging obtain an approximate model for the pdf of the response envelope and phase in the steady state. Compare this with model for the envelope process obtained based on Rice’s approach as applied to the steady state solution of the above equation obtained without applying the averaging approximation.

<u>Monte Carlo simulations</u>

40. Simulate 10000 samples of *X* for the following cases

- $P_X(x)=\left[1-\exp(-2x)\right]^5; 0\le x<\infty$
- $p_X(x)=\frac{1}{\pi}\frac{1}{1+x^2}; -\infty<x<\infty$

In each case, perform the Kolmogorov-Smirnov test to test the hypothesis that the samples are drawn from a population with the respective target PDF-s.

41. Consider the estimator for the variance of a random sample given by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 \text{ with } \bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

Show that the estimator is unbiased and $\text{Var}\left[S^2\right] = \frac{\sigma^4}{n}\left(\frac{\mu^4}{\sigma^4} - \frac{n-3}{n-1}\right)$.

42. By drawing n samples from $U[0,1]$, demonstrate, through numerical experimentation, that the estimator for the mean, given by $\Theta = \frac{1}{n}\sum_{i=1}^{n} X_i$, is unbiased, consistent and has a Gaussian sampling PDF with mean 0.5 and standard deviation $0.2886/\sqrt{n}$.

43. Ten realizations of a lognormally distributed random variable are obtained as 1.8712, 11.6067, 0.8381, 214.0847, 2.0693, 3.4139, 22.9550, 3.0605, 2.2450, and 0.5144. Use the method of maximum likelihood estimation and determine the parameters of the distribution.

44. Simulate 10000 samples of X when X is characterized by the following models

(a) X is a binomial random variable with n=15 and p=0.3
(b) X is a Poisson random variable with parameter $\lambda = 0.4$
In each case, estimate the PDF and graphically compare the function with the target PDF.

45. Use MCMC method and simulate 1000 samples of a random variable R whose pdf is given by

$$p_R(r) = \frac{r}{\sigma_a \sigma_b \sqrt{\left(1 - r_{ab}^2\right)}} \exp\left[-r^2\left(\frac{\sigma_a^2 + \sigma_b^2}{4\sigma_a^2\sigma_b^2\left(1 - r_{ab}^2\right)}\right)\right] I_0\left[r^2\left(\frac{\sqrt{r_{ab}^2 + \left\{\frac{\sigma_a^2 - \sigma_b^2}{2\sigma_a\sigma_b}\right\}^2}}{2\sigma_a\sigma_b\left(1 - r_{ab}^2\right)}\right)\right]$$

$0 < r < \infty; I_0(\bullet) = \text{Bessel's function of the first kind}$

Assume suitable values for the parameters of the pdf. Perform the Kolmogorov-Smirnov test to test the hypothesis that the samples are indeed drawn from a population with underlying pdf as above.

46. Use MCMC method and simulate 1000 samples of a random variable Φ whose pdf is given by

$$p_\Phi(\phi) = \frac{\sqrt{\left(1 - r_{ab}^2\right)}}{2\pi\sigma_a\sigma_b\left[\frac{\cos^2\phi}{\sigma_b^2} + \frac{\sin^2\phi}{\sigma_a^2} - \frac{r_{ab}\sin\phi\cos\phi}{\sigma_a\sigma_b}\right]}; 0 < \phi < 2\pi$$

Assume suitable values for the parameters of the pdf. Perform the Kolmogorov-Smirnov test to test the hypothesis that the samples are indeed drawn from a population with underlying pdf as above.

47. Consider a set of six random variables which have the following properties

X_1	Normal with mean=60 and standard deviation=10
X_2	Lognormal with mean=1 and standard deviation=0.2
X_3	Lognormal with mean=2 and standard deviation=0.1
X_4	Exponetial with parameter λ=1
X_5	Normal with mean=-33 and standard deviation=2
X_6	Normal with mean=3.5 and standard deviation=0.8

$$\rho = \begin{bmatrix} 1.0 & 0.2 & 0 & 0 & -0.4 & 0 \\ 0.2 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 & 1.0 & -0.835 \\ 0 & 0 & 0 & 0 & -0.835 & 1.0 \end{bmatrix}$$

Use Nataf's transformation and simulate 10000 samples of $\{X_i\}_{i=1}^{6}$. Estimate the marginal PDF-s and the covariance matrix and compare them graphically with the respective targets.

48. Consider a zero mean stationary Gaussian random process with PSD function given by

$$S_{XX}(\omega) = \sqrt{\frac{\pi}{\alpha}}\left\{\exp\left(-\frac{(\omega-\beta)^2}{4\alpha}\right) + \exp\left(-\frac{(\omega+\beta)^2}{4\alpha}\right)\right\}$$

Simulate 1000 time histories of the above random process and verify if the target pdf and PSD functions are simulated satisfactorily. Assume suitable values for the parameters in the PSD function.

49. Repeat the exercise above by assuming that the first order pdf of the process is given by a uniformly distributed random variable in -1 to 1.
50. Consider the system in problem 29. Use 1.5 order strong Taylor's expansion and discretize the equation. Simulate 1000 samples of system response and compare the steady state moments with the analytical solutions.
51. Consider the process in problem 23. Using subset simulations, determine the probability that the extreme of the process over a specified duration crosses a level that is 5 times the standard deviation of the process.
52. *X(t)* is a zero mean stationary non-Gaussian random process and it is assumed that the description of the process is limited to the following information:

$$R_{XX}(\tau) = \exp(-\alpha|\tau|); -\infty < \tau < \infty$$

$$p_X(x) = \begin{cases} 0.5 \text{ for } |x| \le 1 \\ 0 \text{ otherwise} \end{cases}$$

Outline procedures for simulation of *X(t)* which employ (a) Fourier series representation, and (b) Karhunen-Loeve expansion. Implement these procedures and simulate 1000 sample sof *X(t)* and verify if the target pdf and autocovariance are satisfactorily realized.

53. X and Y are two non-Gaussian random variables with

$$p_X(x) = \lambda \exp(-\lambda x); 0 < x < \infty$$
$$p_Y(y) = \frac{y}{\sigma^2} \exp\left(-\frac{y}{2\sigma^2}\right); 0 < y < \infty$$

It is also known that the correlation coefficient between X and Y is 0.4. It may be assumed that $\lambda = 1$ and σ=2. Using Nataf's transformation, simulate 10000 samples of X and Y. Verify that the simulation has been performed satisfactorily by using appropriate hypothesis tests.

54. Let $p_i(x_i), i = 1, 2, \cdots, n$ be a set of valid pdf-s and let $a_i \geq 0, i = 1, 2, \cdots, n$ be constants such that $\sum_{i=1}^{n} a_i = 1$. If we now define a function $p_X(x) = \sum_{i=1}^{n} a_i p_i(x)$, it follows that $p_X(x)$ is a valid pdf. Such pdf-s are useful in modeling multi-modal data. Consider the case with

$(a_i)_{i=1}^{5} = [0.1253 \quad 0.2787 \quad 0.0831 \quad 0.3185 \quad 0.1945]$, $p_i(x_i) \sim N(m_i, \sigma_i), i = 1, 2, \cdots, 5$

$(m_i)_{i=1}^{5} = [1 \quad 2.3 \quad 0 \quad -3.1 \quad 3.2]$, and $\sigma_i = 1, i = 1, 2, \cdots, 5$. Simulate 10000 samples of X and compare the estimate of the PDF with the target.