Stochastic Structural Dynamics

Lecture-5

Multi-dimensional random variables-2

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Recall

- Two random variables
  - Joint PDF
  - Joint pdf
  - Conditional PDF
  - Conditional pdf & Conditional expectations
  - Independence of RVs
  - Joint Expectations
- Correlation function
- Functions of random variables

Example

Given $R = \sqrt{X^2 + Y^2}$ & $\Theta = \tan^{-1}\left(\dfrac{Y}{X}\right)$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Find $p_{R\Theta}(r, \theta)$.

- $p_{XY}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right); -\infty < x, y < \infty$
- $0 < r < \infty; \quad 0 < \theta < 2\pi$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$J^{-1} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Joint pdf

$$p_{R\Theta}(r,\theta) = \left.\frac{p_{XY}(x,y)}{|J|}\right|_{\substack{x=r\cos\theta \\ y=r\sin\theta}}$$

$$= \frac{r}{2\pi}\exp\left[-\frac{1}{2}\left(r^2\cos^2 + r^2\sin^2\theta\right)\right]$$

$$= \frac{r}{2\pi}\exp\left[-\frac{r^2}{2}\right]; \quad 0 < r < \infty; 0 < \theta < 2\pi$$

Marginal pdf

$$p_R(r) = \int_0^{2\pi} p_{R\Theta}(r,\theta)d\theta$$

$$= \int_0^{2\pi} \frac{r}{2\pi}\exp\left[-\frac{r^2}{2}\right]d\theta$$

$$= r\exp\left[-\frac{r^2}{2}\right]; \quad 0 < r < \infty$$

$$p_\Theta(\theta) = \int_0^{\infty} p_{R\Theta}(r,\theta)dr$$

$$= \int_0^{\infty} \frac{r}{2\pi}\exp\left[-\frac{r^2}{2}\right]dr$$

$$= \frac{1}{2\pi}; \quad 0 < \theta < 2\pi$$

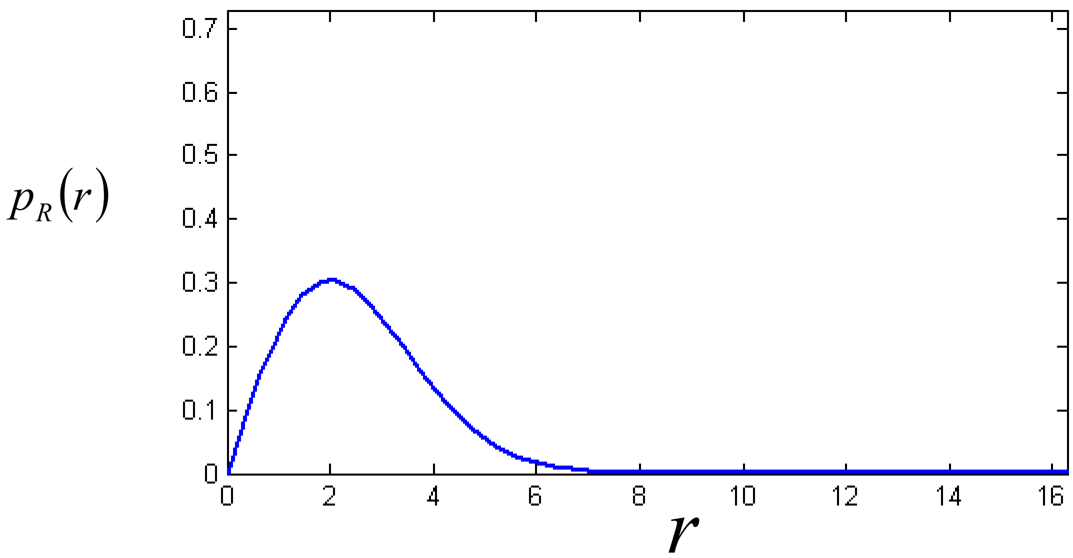
$$p(r,\theta) = p_R(r)p_\Theta(\theta)$$

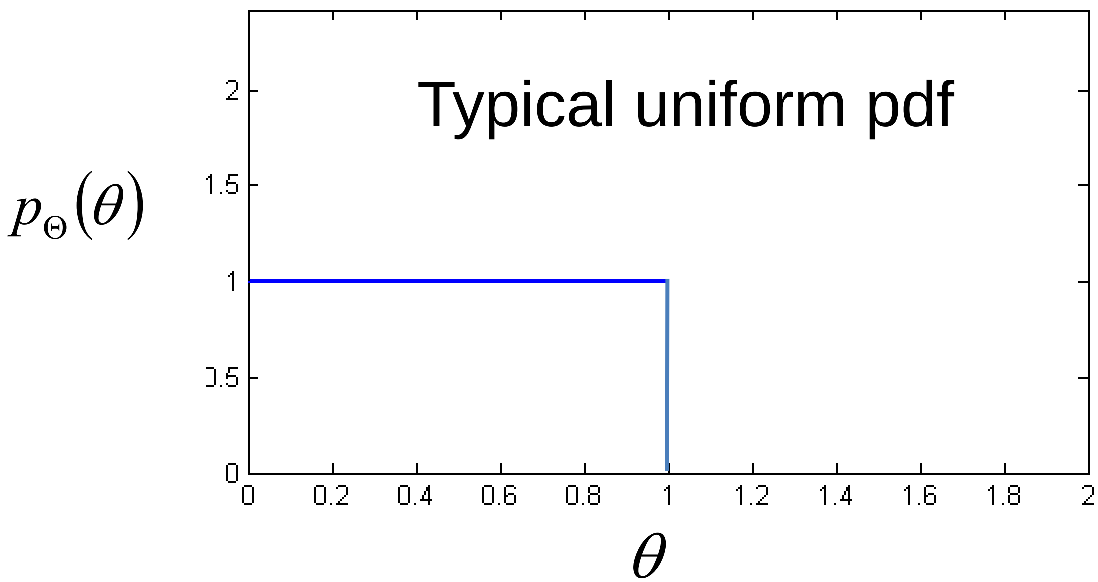
$$\Rightarrow R \perp \Theta$$

R: Rayleigh RV

Θ: Uniformly distributed RV

Typical Rayleigh pdf





Example (Box - Muller Transformation)

Let X and Y be independent, and uniformly distributed random variables in 0 to 1. Define

$$U = \left(-2\ln X\right)^{\frac{1}{2}} \cos\left(2\pi Y\right)$$

$$V = \left(-2\ln X\right)^{\frac{1}{2}} \sin\left(2\pi Y\right)$$

Deptermine $p_{UV}\left(u, v\right)$.

Box-Muller Transformation (continued)

$$U = \left(-2\ln X\right)^{\frac{1}{2}} \cos\left(2\pi Y\right)$$

$$V = \left(-2\ln X\right)^{\frac{1}{2}} \sin\left(2\pi Y\right)$$

$$\Rightarrow u^2 + v^2 = \left(-2\ln x\right) \Rightarrow x = \exp\left(-\frac{u^2 + v^2}{2}\right)$$

$$\frac{v}{u} = \tan\left(2\pi y\right) \Rightarrow y = \frac{1}{2\pi}\tan^{-1}\left(\frac{v}{u}\right)$$

Box-Muller Transformation (continued)

$$J^{-1} = \begin{vmatrix} \exp\left(-\frac{u^2+v^2}{2}\right) & \exp\left(-\frac{u^2+v^2}{2}\right)(-v) \\ \frac{1}{2\pi}\frac{1}{1+\left(\frac{v}{u}\right)^2}\left(-\frac{v^2}{u}\right) & \frac{1}{2\pi}\frac{1}{1+\left(\frac{v}{u}\right)^2}\left(\frac{1}{u}\right) \end{vmatrix} = -\frac{1}{2\pi}\exp\left(-\frac{u^2+v^2}{2}\right)$$

$$p_{UV}(u,v) = \frac{1}{2\pi}\exp\left(-\frac{u^2+v^2}{2}\right); -\infty < u, v < \infty$$

$$\Rightarrow U \perp V \;\&\; U \sim N(0,1) \;\&\; V \sim N(0,1)$$

This transformation could be used in simulation of Gaussian random numbers on a computer (more on this later).

Example

Given $U = X^2 + Y^2$ & $V = \frac{X}{Y}$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

Determine $p_{UV}(u, v)$ and check if U and V are independent.

$$U = X^2 + Y^2 \;\&\; V = \frac{X}{Y}$$

$$\Rightarrow 0 < u < \infty \;\&\; -\infty < v < \infty$$

$$x = vy$$

$$u = y^2\left(1 + v^2\right)$$

$$\Rightarrow y^2 = \frac{u}{\left(1 + v^2\right)} \;\&\; x^2 = \frac{uv^2}{\left(1 + v^2\right)}$$

$$J = \begin{vmatrix} 2x & 2y \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -2\left(1 + \frac{x^2}{y^2}\right) = -2\left(1 + v^2\right)$$

$$p_{UV}(u,v) = \frac{1}{2\pi\sigma^2}\frac{1}{2(1+v^2)}\exp\left(-\frac{u}{2\sigma^2}\right);$$

with $0 < u < \infty, -\infty < v < \infty$

$$\Rightarrow p_U(u) = \int_{-\infty}^{\infty} p_{UV}(u,v)dv = \frac{1}{2\sigma^2}\exp\left(-\frac{u}{2\sigma^2}\right); 0 < u < \infty$$

$$\& \ p_V(v) = \int_{0}^{\infty} p_{UV}(u,v)du = \frac{1}{2\pi(1+v^2)}; -\infty < v < \infty$$

$$\Rightarrow p_{UV}(u,v) = p_U(u)p_V(v)$$

$$\Rightarrow U \perp V$$

U is exponetially distributed and V is Cauchy distributed.

Example

Given $U = X + Y$ & $\begin{pmatrix} X \\ Y \end{pmatrix} \sim p_{XY}(x, y)$

Determine $p_U(u)$.

Note: Here we are dealing with one function of two random variables.

Strategy

Introduce a dummy variable.

$$U = X + Y$$

$$V = Y$$

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$p_{UV}(u, v) = p_{XY}(x, y)\Big|_{\substack{x=u-v \\ y=v}}$$

$$p_U(u) = \int_{-\infty}^{\infty} p_{XY}(u - v, v)\,dv$$

Note: If $X \perp Y \Rightarrow$

$$p_U(u) = \int_{-\infty}^{\infty} p_X(u - v)\,p_Y(v)\,dv$$

Note: The above integral represents the convolution operation.

Example

Let $p_X(x) = a\exp(-ax)$ & $p_Y(y) = b\exp(-by)$

with $0 < x,y < \infty$.

Define $U = X + Y$. Find $p_U(u)$.

Solution

Introduce the dummy variable $V = Y$.

$$p_U(u) = \int_{-\infty}^{\infty} p_X(u-v)\,p_Y(v)\,dv$$

$$p_U(u) = \int_{-\infty}^{\infty} p_X(u-v)\,p_Y(v)\,dv$$

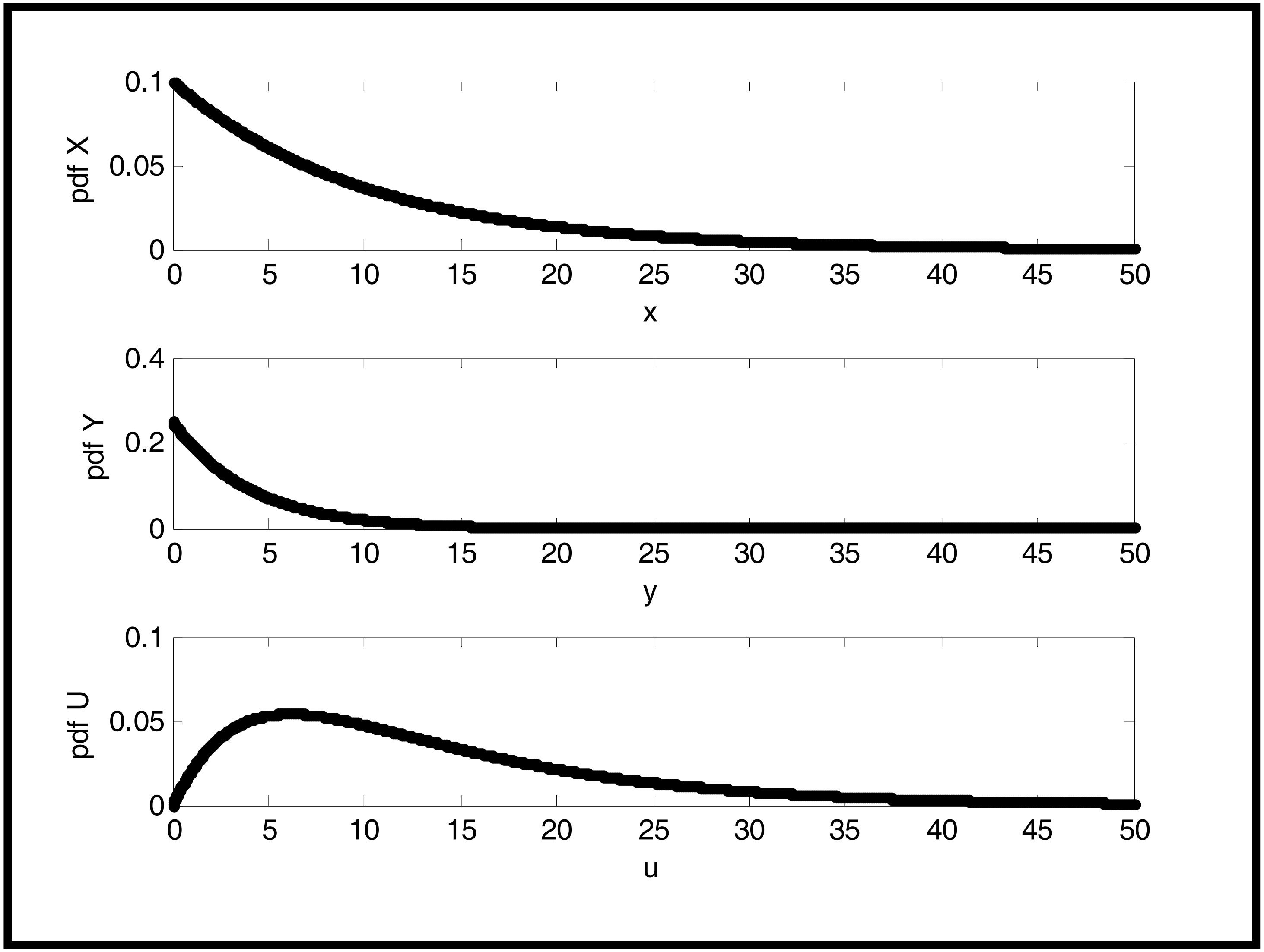
$$= \int_{0}^{\infty} b\exp(-bv)\,p_X(u-v)\,dv$$

Since $p_X(u-v) = 0 \; \forall u - v < 0 \Rightarrow$

$$p_U(u) = \int_{0}^{u} b\exp(-bv)\,p_X(u-v)\,dv$$

$$= \int_{0}^{u} b\exp(-bv)\,a\exp(-a(u-v))\,dv$$

$$= \frac{ab}{b-a}\left[\exp(-au) - \exp(-bv)\right]; 0 < u < \infty$$



0.1
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pdf X
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x
0.4
0.2
0
pdf Y
0 5 10 15 20 25 30 35 40 45 50
y
0.1
0.05
0
pdf U
0 5 10 15 20 25 30 35 40 45 50
u

More than one random variables

Let $\tilde{X} = \{X_i\}_{i=1}^{n}$ be a set of random variables.

Definitions

n - th order Joint PDF of $\tilde{X}$

$$P_{\tilde{X}}(\tilde{x}) = P\left(\bigcap_{i=1}^{n}(X_i \le x_i)\right) \text{ where } \tilde{x} = \{x_i\}_{i=1}^{n}.$$

n **- th order Joint pdf of** $\tilde{X}$

$$p_{\tilde{X}}(\tilde{x}) = \frac{\partial^n P_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}$$

Expectation of $g(\tilde{X})$

$$\left\langle g(\tilde{X}) \right\rangle = \int_{-\infty}^{\infty} g(\tilde{x}) p_{\tilde{X}}(\tilde{x}) d\tilde{x}$$

Note : The above is an n **- fold integral.**

Multi - dimensional Gaussian random variable

Let $X = \{X_1 \quad X_2 \quad \cdots \quad X_n\}^t$ be a vector of random variables.

Let $m_i = \langle X_i \rangle$ such that the mean vector is given by

$m = \{m_1 \quad m_2 \quad \cdots \quad m_n\}^t$.

The elements of covariance matrix C is given by

$C_{ij} = \left\langle \left(X_i - m_i\right)\left(X_j - m_j\right)\right\rangle$; clearly $C_{ij} = C_{ji}$ and hence $C^t = C$.

Assume that C^{-1} exists.

Let $x = \{x_1 \quad x_2 \quad \cdots \quad x_n\}^t$ be a realization of $X = \{X_1 \quad X_2 \quad \cdots \quad X_n\}^t$.

Definition

$X = \{X_1 \quad X_2 \quad \cdots \quad X_n\}^t$ is said to be Gaussian distributed if

$$p_X(x) = p_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |C|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left\{(x-m)^t C^{-1} (x-m)\right\}\right]; -\infty < x_i < \infty \, \forall i = 1, 2, \cdots, n$$

Remarks

(1) $n = 1$

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma} \right)^2 \right); -\infty < x < \infty$$

(2) $n = 2 \Rightarrow X = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \quad m = \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} \quad x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

$$C = \begin{bmatrix} \langle (X_1 - m_1)^2 \rangle & \langle (X_1 - m_1)(X_2 - m_2) \rangle \\ \langle (X_1 - m_1)(X_2 - m_2) \rangle & \langle (X_2 - m_2)^2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\Rightarrow |C| = (1-\rho^2)\sigma_1^2\sigma_2^2$$

$$C^{-1} = \begin{bmatrix} \dfrac{1}{(1-\rho^2)\sigma_1^2} & \dfrac{-\rho}{(1-\rho^2)\sigma_1\sigma_2} \\ \dfrac{-\rho}{(1-\rho^2)\sigma_1\sigma_2} & (1-\rho^2)\sigma_2^2 \end{bmatrix}$$

$$\Rightarrow p_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} \exp\left[-\frac{1}{2(1-\rho^2)} \left\{ \frac{(x_1-m_1)^2}{\sigma_1^2} + \frac{(x_2-m_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2} \right\} \right]$$

$-\infty < x_1, x_2 < \infty$

Given a set of correlated Gaussian random variables, how to transform them into a set of uncorrelated Gaussian random variables?

$\left\{X_i\right\}_{i=1}^{n}$ = correlated Gaussian rvs.

$$\left\langle X_i \right\rangle = \mu_i; C_{ij} = \left\langle \left(X_i - \mu_i \right)\left(X_j - \mu_j \right) \right\rangle$$

$$X_i' = \frac{X_i - \mu_i}{\sigma_i} \Rightarrow \left\langle X_i' \right\rangle = 0$$

$$C_{ij}' = \left\langle X_i' X_j' \right\rangle = \begin{bmatrix} 1 & C_{12}' & C_{13}' \cdots & C_{1n}' \\ C_{21}' & 1 & C_{23}' \cdots & C_{2n}' \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1}' & C_{n2}' & C_{n3}' \cdots & 1 \end{bmatrix}$$

Transform

$$Y = T^t X' \Rightarrow \left\langle Y \right\rangle = 0; \left\langle Y Y^t \right\rangle = \left\langle T^t X' X'^t T \right\rangle = T^t C' T$$

Select T such that $T^t C' T = I$

How to select T?

Consider the eigenvalue problem

$$[C']\{\alpha\} = \lambda\{\alpha\}$$

eigenvalues: $|C' - I\lambda| = 0$

$\Rightarrow n$ eigenvalues $\{\lambda_i\}_{i=1}^{n}$

C' is positive definite $\Rightarrow$

$\lambda_i > 0 \forall i = 1, 2, \cdots, n$

eigenvectors : Φ.

$$C'\phi_i = \lambda_i \phi_i$$

$$C'\phi_j = \lambda_j \phi_j$$

$$\phi_j^t C' \phi_i = \lambda_i \phi_j^t \phi_i$$

$$\phi_i^t C' \phi_j = \lambda_j \phi_i^t \phi_j$$

$$\phi_j^t C' \phi_i = \lambda_j \phi_j^t \phi_i \quad \left(\because C' = C'^t\right)$$

$$\Rightarrow (\lambda_i - \lambda_j)\phi_j^t \phi_i = 0 \Rightarrow \phi_j^t \phi_i = 0 \forall i \neq j$$

$$\Rightarrow \phi_i^t C' \phi_j = 0 \forall i \neq j$$

Select Φ such that $\Phi^t C' \Phi = I$.

Take $T = \Phi$.

Given a Gaussian random variable, how to transform it into a specified non-Gaussian random variable?

Example
X be a
Rayleigh
RV

$Y = g(x)$

$$p_X(x) = \frac{x}{\sigma^2}\exp\left[-\frac{x^2}{2\sigma^2}\right]; 0 \le x < \infty$$

$$P_X(x) = \int_0^x \frac{s}{\sigma^2}\exp\left[-\frac{s^2}{2\sigma^2}\right]ds = 1 - \exp\left[-\frac{x^2}{2\sigma^2}\right]$$

Let $Z \sim N(0,1)$ and consider the transformation

$$1 - \exp\left[-\frac{X^2}{2\sigma^2}\right] = \Phi[Z]$$

with $\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp\left[-\frac{s^2}{2}\right]ds$

$$\Rightarrow \exp\left[-\frac{X^2}{2\sigma^2}\right] = 1 - \Phi[Z]$$

$$\Rightarrow X = \sqrt{-2\sigma^2 \log\left[1 - \Phi[Z]\right]}$$

Claim : X is Rayleigh

$$1-\exp\left[-\frac{x^2}{2\sigma^2}\right]=\Phi(z)$$

$$\Rightarrow \frac{x}{\sigma^2}\exp\left[-\frac{x^2}{2\sigma^2}\right]\frac{dx}{dz}=\phi(z)$$

$$\Rightarrow \frac{dx}{dz}=\frac{\phi(z)}{\frac{x}{\sigma^2}\exp\left[-\frac{x^2}{2\sigma^2}\right]}$$

$$\Rightarrow p_X(x)=\frac{\phi(z)}{\left|\frac{dx}{dz}\right|}$$

$$=\frac{x}{\sigma^2}\exp\left[-\frac{x^2}{2\sigma^2}\right];0\le x<\infty;QED$$

$$\phi(z)=\frac{1}{2\pi}e^{-\frac{z^2}{2}}$$

$$-\infty<z<\infty$$

In general: Let X be a RV with PDF $P_X(x)$.

Consider

$$P_X(X) = \Phi(Z) \Rightarrow X = P_X^{-1}\left[\Phi(Z)\right]$$

$$\Rightarrow p_X(x)\frac{dx}{dz} = \phi(z)$$

$$\Rightarrow p_X(x) = \frac{\phi(z)}{|\frac{dx}{dz}|} = p_X(x).$$

Rosenblatt transformation

Case of two random variables

Let X_1 & X_2 be two non-Gaussian RVs.

$$\text{JPDF}: P_{12}(x_1, x_2)$$

$$\text{jpdf}: p_{12}(x_1, x_2)$$

$$\text{MPDF}: P_1(x_1) \,\&\, P_2(x_2)$$

$$\text{mpdf}: p_1(x_1) \,\&\, p_2(x_2)$$

Let $U_1 \rightarrow N(0,1)$ & $U_2 \rightarrow N(0,1)$ with $U_1 \perp U_2$

Define

$$P_1(X_1) = \Phi(U_1)$$

$$P_2(X_2 \mid X_1) = \Phi(U_2)$$

$$p_1(x_1)\frac{dx_1}{du_1} = \phi(u_1); \frac{dx_1}{du_2} = 0$$

$$p_2(x_2 \mid x_1)\frac{dx_2}{du_2} = \phi(u_2)$$

$$\Rightarrow p_{12}(x_1, x_2) = \frac{\phi(u_1, u_2)}{|J|} @ u_1$$

$$= \Phi^{-1}\{P_1(x_1)\} \,\&\, u_2 = \Phi^{-1}\{P_2(x_2 \mid x_1)\}$$

$$\Rightarrow p_{12}(x_1, x_2) =$$

$$\frac{\phi(u_1)\phi(u_2)}{\phi(u_1)\phi(u_2)} p_2(x_2 \mid x_1) p_1(x_1) = p_{12}(x_1, x_2)$$

Models for limits of sums, products and extremes

Central limit theorem

Let $\left\{X_i\right\}_{i=1}^{n}$ be an iid sequence (iid=independent, identically distributed) of random variables.

Let $\left\langle \mathrm{X_i} \right\rangle = \mu \;\&\; \left\langle \left(X_i - \mu\right)^2 \right\rangle = \sigma^2.$

According to the central limit theorem

$$\lim_{n\to\infty} P\left[\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx = \Phi\left(a\right)$$

Central limit theorem for products

Let $\left\{X_i\right\}_{i=1}^{n}$ be an iid sequence (iid=independent, identically distributed) of random variables that take only positive values.

Define

$$Y = X_1 X_2 \cdots X_n$$

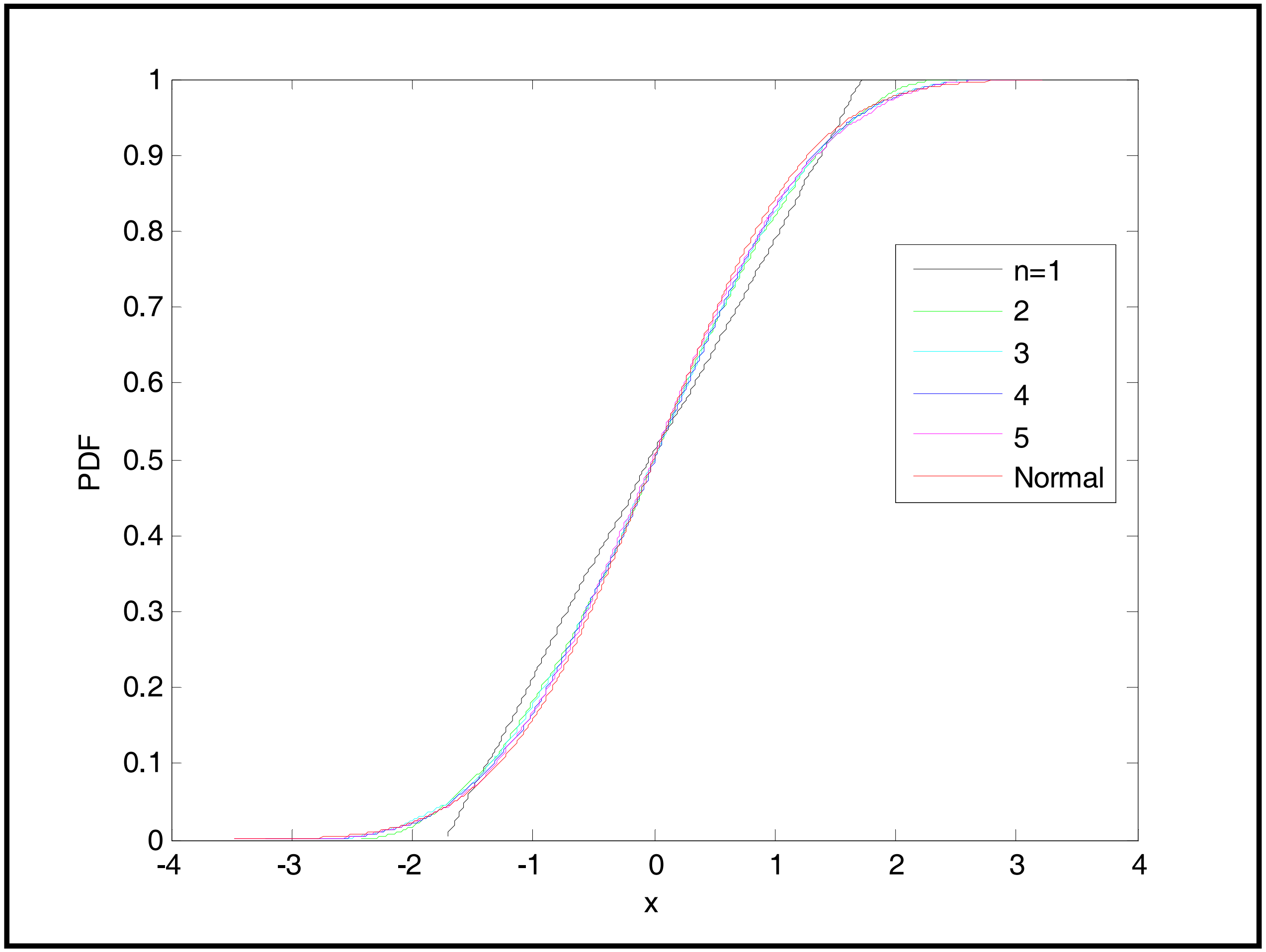
$$\lim_{n\to\infty} p_Y\left(y\right) \to \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}\left(\ln y - \eta\right)^2\right) U\left(y\right)$$

with $\eta = \sum_{i=1}^{n} \left\langle \ln X_i \right\rangle \;\&\; \sigma^2 = \sum_{i=1}^{n} \mathrm{Var}\left(\ln X_i\right)$

Example

$\{X_i\}_{i=1}^{n}$ are iid, uniformly distributed in 0 to 1.

$$\mathrm{Sum}_n = \frac{X_1 + X_2 + \cdots + X_n - \mu n}{\sigma\sqrt{n}}$$



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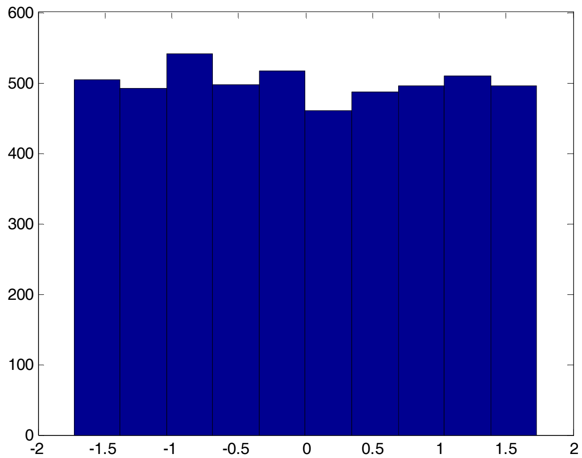
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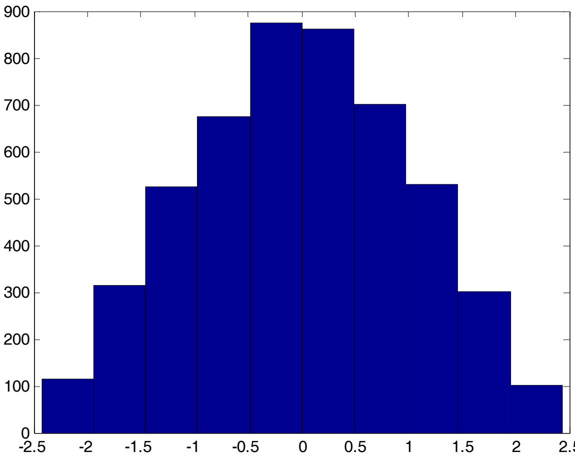
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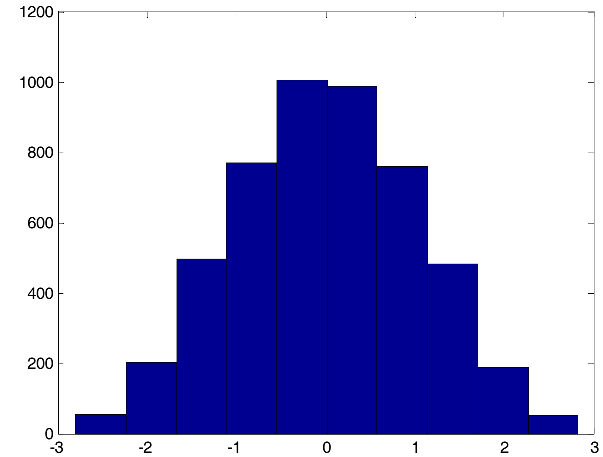
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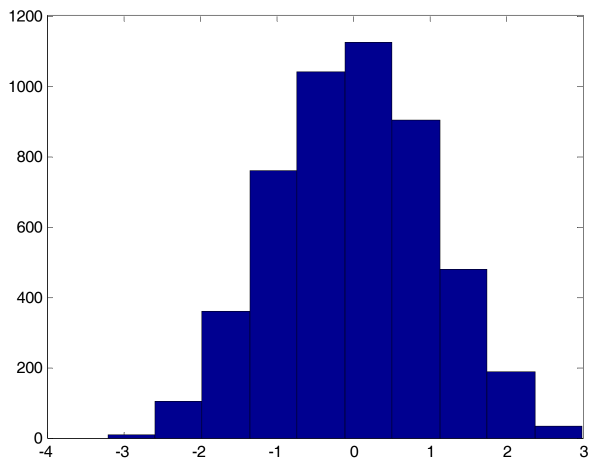
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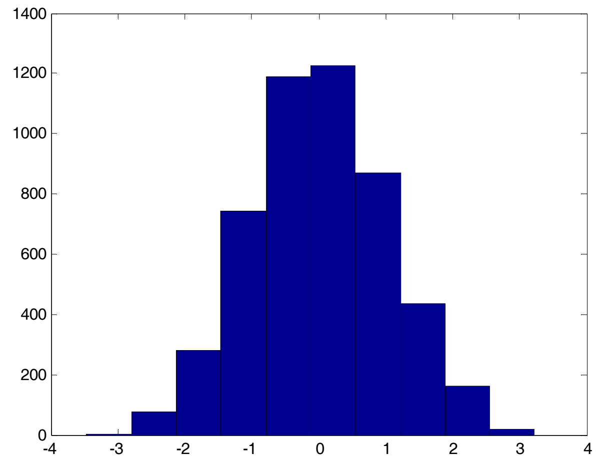
N=5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-4 -3 -2 -1 0 1 2 3 4

Extreme value distributions: distribution of maxima

Example

Let X and Y be two random variables with specified jpdf.

Define Z=max(X,Y). What is the pdf of Z?

$$P_Z\left(z\right)=P(Z\le z)$$

$$=P\left(\max\left\{X,Y\right\}\le z\right)$$

$$=P\left(X\le z\cap Y\le z\right)=P_{XY}\left(z,z\right)$$

$$=P_X\left(z\right)P_Y\left(z\right)\ (\text{if X}\perp\text{Y})$$

$$=\left[P_X\left(z\right)\right]^2\ (\text{if X and Y are iid}).$$

$$\mathrm{p}_\mathrm{Z}\left(z\right)=2P_X\left(z\right)p_X\left(z\right)$$

Generalization

Let $\left\{X_i\right\}_{i=1}^{n}$ be an iid sequence.

Define $Z = \max\left\{X_i\right\}_{i=1}^{n}$.

$$P_Z(z) = \left[P_X(z)\right]^n$$

$$p_Z(z) = n\left[P_X(z)\right]^{n-1} p_X(z)$$

Extreme value distributions: distribution of minima

Example

Let X and Y be two random variables with specified jpdf.

Define Z=min(X,Y). What is the pdf of Z?

$$P_Z(z) = P(Z > z)$$

$$= P\left(\min\{X,Y\} > z\right)$$

$$= \{1 - P_X(z)\}\{1 - P_Y(z)\} \text{ (if } X \perp Y\text{)}$$

$$= \{1 - P_X(z)\}^2 \text{ (if } X \text{ and } Y \text{ are iid)}$$

$$1 - P_Z(z) = \{1 - P_X(z)\}^2$$

$$\Rightarrow P_Z(z) = 1 - \{1 - P_X(z)\}^2$$

$$\Rightarrow p_Z(z) = 2\{1 - P_X(z)\} p_X(z)$$

Generalization

Let $\left\{X_i\right\}_{i=1}^{n}$ be an iid sequence.

Define $Z = \min\left\{X_i\right\}_{i=1}^{n}$.

$$P_Z(z) = 1 - \left[1 - P_X(z)\right]^n$$

$$p_Z(z) = n\left[1 - P_X(z)\right]^{n-1} p_X(z)$$

Asymptotic extreme value distributions

Let $\{X_i\}_{i=1}^{n}$ be an iid sequence.

Define

$$Z = \max\{X_i\}_{i=1}^{n}$$

$$Y = \min\{X_i\}_{i=1}^{n}$$

We have shown that

$$P_Z(z) = \left[P_X(z)\right]^n$$

$$P_Y(y) = 1 - \left[1 - P_X(y)\right]^n.$$

Questions:

what happens as $n \to \infty$?

$P_X(x)$ is not known?

n is not known?

Degeneracy

$$\lim_{n\to\infty}\left[P_X(z)\right]^n \to 1 \text{ if } P_X(z) = 1$$

$$\lim_{n\to\infty}\left[P_X(z)\right]^n \to 0 \text{ if } P_X(z) < 1$$

$$\lim_{n\to\infty}\left\{1 - \left[1 - P_X(z)\right]^n\right\} \to 0 \text{ if } P_X(z) = 0$$

$$\lim_{n\to\infty}\left\{1 - \left[1 - P_X(z)\right]^n\right\} \to 1 \text{ if } 0 < P_X(z) \le 1$$

Stability hypothesis

$$F^n(x) = F(a_n x + b_n)$$

Tail decaying exponentially $\Rightarrow$ Type I (Gumbel)

Tail decaying as a polynomial $\Rightarrow$ Type II (Frechet)

Tail is bounded $\Rightarrow$ Type III (Weibull)

Maxima

$H_{1,\gamma}(x) = \exp(-x^{-\gamma})$	if $x>0$
	otherwise
$H_{2,\gamma}(x) = 1$	if $x \geq 0$
$= \exp\left[-(-x)^{\gamma}\right]$	otherwise
$H_{3,0}(x) = \exp\left[-\exp(-x)\right]$	$-\infty < x < \infty$

Minima

$$L_{1,\gamma}(x) = 1 - \exp\left[-(-x)^{-\gamma}\right] \quad \text{if } x<0$$

$$= 1 \quad \text{otherwise}$$

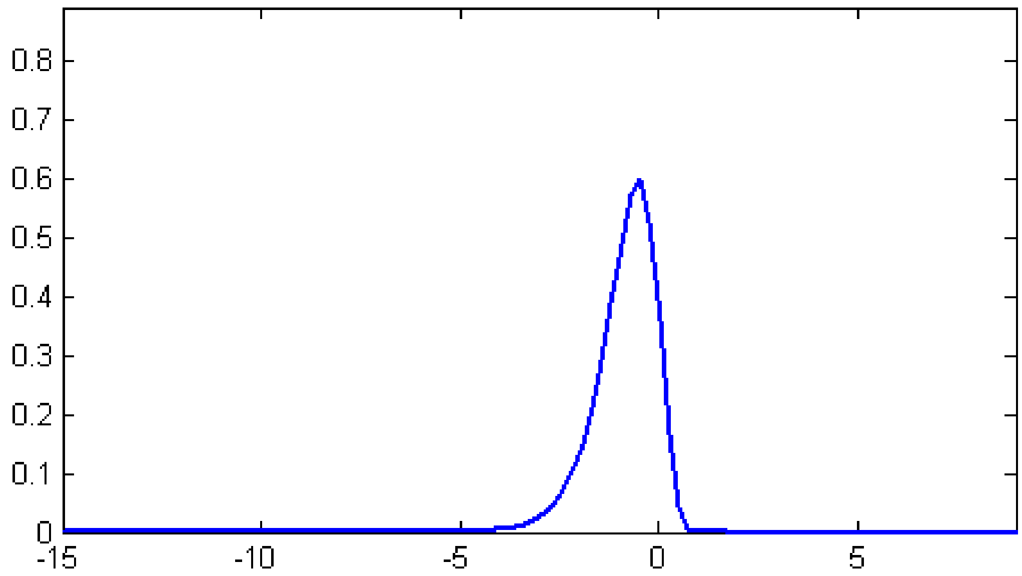
$$L_{2,\gamma}(x) = 1 - \exp(-x^{-\gamma}) \quad \text{if } x > 0$$

$$= 0 \quad \text{otherwise}$$

$$L_{3,0}(x) = 1 - \exp\left[-\exp(-x)\right] \quad -\infty < x < \infty$$

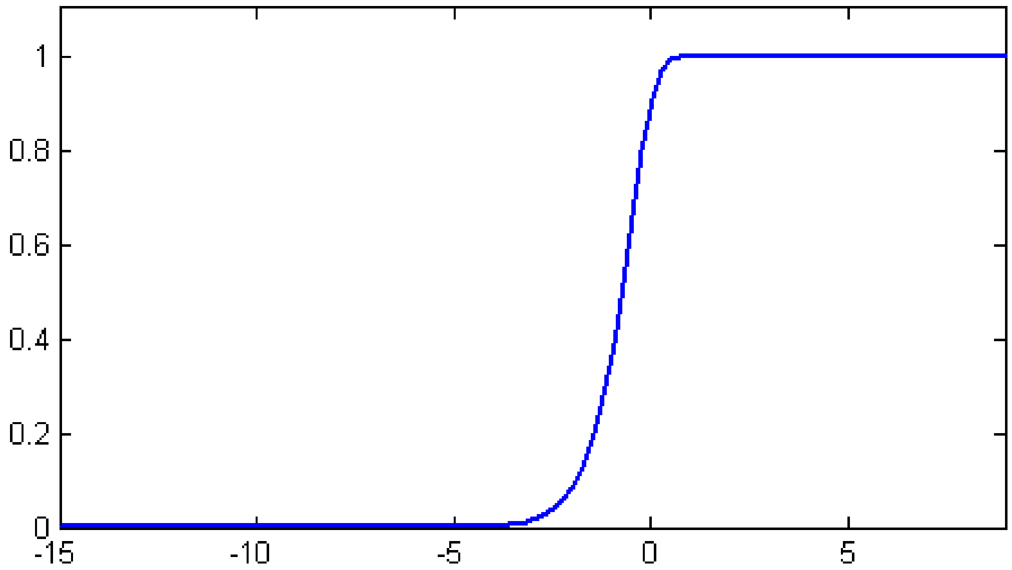
Gumbel RV

pdf



x

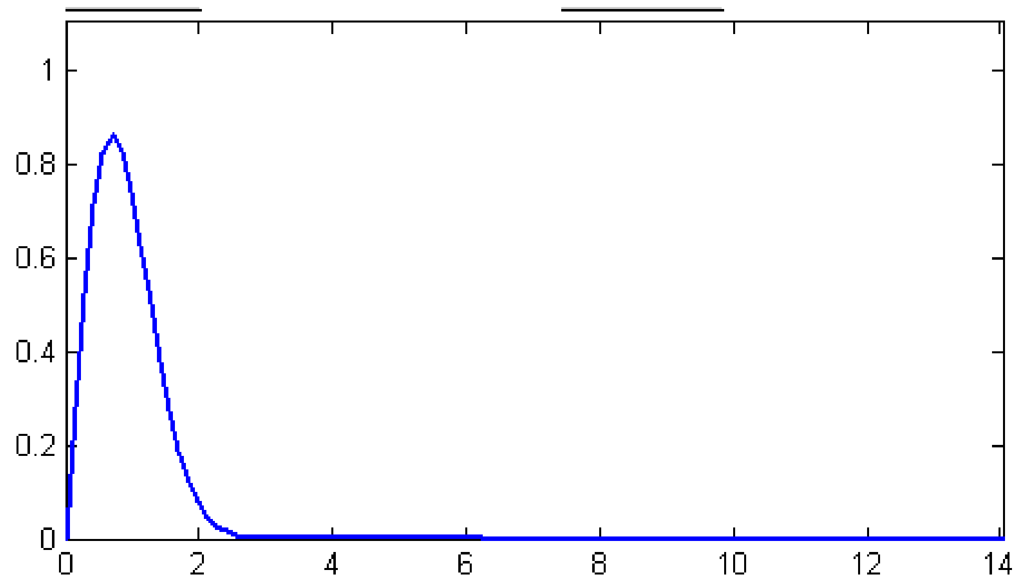
PDF



x

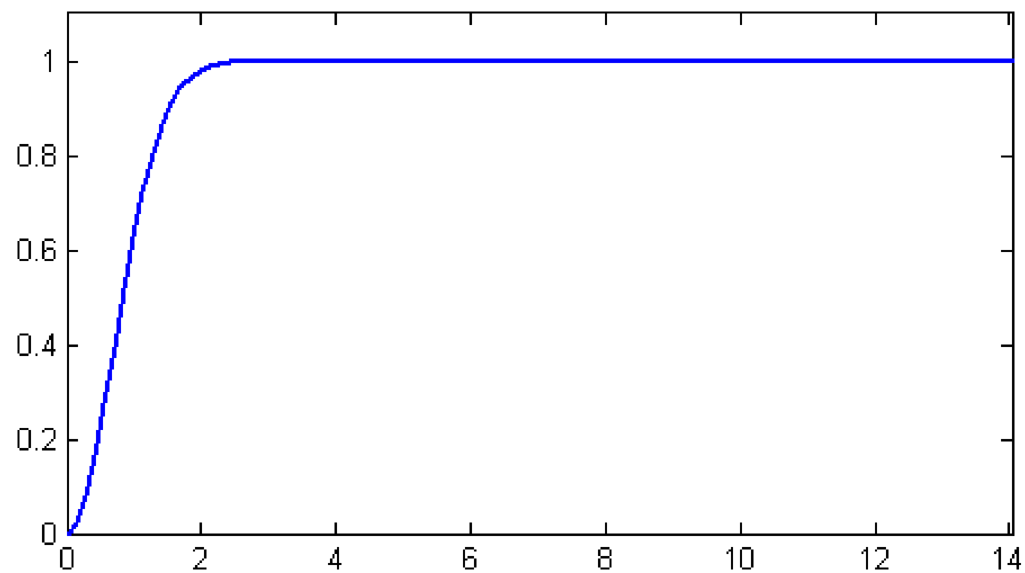
Weibull RV

pdf



x

PDF



x

Domains of attraction of a few commonly occuring Random variables

RV	Domain of attraction for maxima	Domain of attraction for minima
Gaussian	Gumbel	Gumbel
Exponential	Gumbel	Weibull
Cauchy	Frechet	Frechet
Rayleigh	Gumbel	Gumbel
Log-normal	Gumbel	Gumbel
Unifrom	Weibull	Weibull

Table 1 Properties of Gumbel distributions

Quantity	Gumbel (maximum)	Gumbel (minimum)
PDF	$G(x) = \exp\left[-\exp\left\{\frac{-(x-\lambda))}{\delta}\right\}\right]$	$\overline{G}(x) = 1-\exp\left[-\exp\left(\frac{-(\lambda-x)}{\delta}\right)\right]$
Mean	$\lambda + 0.57772\delta$	$\lambda - 0.57772\delta$
Median	$\lambda + 0.3665\delta$	$\lambda - 0.3665\delta$
Mode	λ	λ
Variance	$\frac{\pi^2\delta^2}{6}$	$\frac{\pi^2\delta^2}{6}$
Kurtosis coefficient	1.1396	-1.1396

Table 2 Properties of Weibull distributions

Quantity	Weibull (maximum)	Weibull (minimum)
PDF	$G(x) = \exp\left[-\left(\frac{\lambda - x}{\delta}\right)^{\beta}\right]$	$G(x) = 1 - \exp\left[-\left(\frac{\lambda - x}{\delta}\right)^{\beta}\right]$
Mean	$\lambda - \delta\Gamma\left(1 + \frac{1}{\beta}\right)$	$\lambda + \delta\Gamma\left(1 + \frac{1}{\beta}\right)$
Median	$\lambda - \delta 0.693^{\frac{1}{\beta}}$	$\lambda + \delta 0.693^{\frac{1}{\beta}}$
Mode	$\lambda - \delta\left(\frac{\beta - 1}{\beta}\right)^{\frac{1}{\beta}}; \beta > 1$ $\lambda \quad ; \beta \le 1$	$\lambda + \delta\left(\frac{\beta - 1}{\beta}\right)^{\frac{1}{\beta}}; \beta > 1$ $\lambda \quad ; \beta \le 1$
Variance	$\delta^2\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right]$	$\delta^2\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right]$

Table 3 Properties of Frechet distributions

Quantity	Weibull (maximum)	Weibull (minimum)
PDF	$G(x) = \exp\left[-\left(\frac{x}{x-\lambda}\right)^{\beta}\right]$	$G(x) = 1 - \exp\left[-\left(\frac{x}{x-\lambda}\right)^{\beta}\right]$
Mean	$\lambda + \delta\Gamma\left(1-\frac{1}{\beta}\right); \beta > 1$	$\lambda - \delta\Gamma\left(1-\frac{1}{\beta}\right); \beta > 1$
Median	$\lambda + \delta 0.693^{-\frac{1}{\beta}}$	$\lambda - \delta 0.693^{-\frac{1}{\beta}}$
Variance	$\delta^2\left[\Gamma\left(1-\frac{2}{\beta}\right) - \Gamma^2\left(1-\frac{1}{\beta}\right)\right]; \beta > 2$	$\delta^2\left[\Gamma\left(1-\frac{2}{\beta}\right) - \Gamma^2\left(1-\frac{1}{\beta}\right)\right]; \beta > 2$

Bayes' theorem

Let X be a random variable and A be an event.
Let us take that A is observable and X is not.
Before observing A we have a model for X.
How can we update the model for X after we have observed A?

Recall

$$
\begin{aligned}
P\left(A \mid x < X \leq x + dx\right) &= \frac{P\left(A \cap x < X \leq x + dx\right)}{P\left(x < X \leq x + dx\right)} \\
&= \frac{P\left(x < X \leq x + dx \mid A\right) P\left(A\right)}{P\left(x < X \leq x + dx\right)} \\
&= \frac{p_X\left(x \mid A\right) P\left(A\right)}{p_X\left(x\right)}
\end{aligned}
$$

$$\lim \quad dx \to 0 \Rightarrow$$

$$P(A \mid X = x) = \frac{p_X(x \mid A) P(A)}{p_X(x)} \Rightarrow$$

$$p_X(x \mid A) = \frac{P(A \mid X = x) p_X(x)}{P(A)} \Rightarrow$$

$$P(A \mid X = x) p_X(x) = p_X(x \mid A) P(A) \Rightarrow$$

$$\int_{-\infty}^{\infty} P(A \mid X = x) p_X(x) dx = \int_{-\infty}^{\infty} p_X(x \mid A) P(A) dx \Rightarrow$$

$$P(A) = \frac{\int_{-\infty}^{\infty} P(A \mid X = x) p_X(x) dx}{\int_{-\infty}^{\infty} p_X(x \mid A) dx} = \int_{-\infty}^{\infty} P(A \mid X = x) p_X(x) dx \Rightarrow$$

$$p_X(x \mid A) = \frac{P(A \mid X = x) p_X(x)}{\int_{-\infty}^{\infty} P(A \mid X = x) p_X(x) dx}$$

Bayes' theorem

$$p_X(x \mid A) = N P(A \mid X = x)\, p_X(x)$$

Remarks

If $P(A \mid X = x) = P(A) \Rightarrow p_X(x \mid A) = N P(A)\, p_X(x)$

$\Rightarrow p(x \mid A) = p_X(x)$

If the objective of observing *A* is to get an improved estimate of pdf of *X*, then *A* and $[x < X \le x + dx]$ should not be independent

Interpretation of *P(A|X=x)*

Let *Y* be a random variable that is observable and dependent on *X*.
Let $y_1, y_2, \cdots, y_n$ be the observations made on Y.

$$P(A \mid X = x) = \prod_{i=1}^{n} p_Y(y_i \mid X = x) \quad (\text{likelihood function})$$

$$p_X^*(x) = NL(y_{1:n} \mid X = x)\, p_X(x)$$

$p_X^*(x) = \text{Posterior pdf}$

$p_X(x) = \text{Apriori pdf}$

L=Likelihood function

N=Normalization constant