Stochastic Structural Dynamics

Lecture-4

Multi-dimensional random variables-1

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Recall

Transformation of random variables

Let *X* be RV; define *Y=g(X);*
Given pdf of *X*, what is the pdf of *Y*?

Mathematical expectation operator

$$\langle g(X)\rangle = E[g(X)] = \int_{-\infty}^{\infty} g(x)\, p_X(x)\, dx$$

Mean, variance, standard deviation
COV, skewness, kurtosis
Characteristic function
Moment generating function

Complete specification of a RV

- Specification of the probability space.
- PDF
- pdf
- Moment generating function
- Characteristic function
- Moments of all orders

Poisson random variable

$$P(X=k)=\exp(-a)\frac{a^k}{k!};k=0,1,2,\cdots,$$

$$\sum_{k=0}^{\infty}\exp(-a)\frac{a^k}{k!}=\exp(-a)\exp(a)=1$$

$$\langle X\rangle=\sum_{k=0}^{\infty}k\exp(-a)\frac{a^k}{k!}=\sum_{k=1}^{\infty}k\exp(-a)\frac{a^k}{k!}$$

$$=\sum_{k=1}^{\infty}k\exp(-a)a\frac{a^{k-1}}{k(k-1)!}$$

$$=a\sum_{k=1}^{\infty}\exp(-a)\frac{a^{k-1}}{(k-1)!}\quad(put\ n=k\text{-}1)$$

$$=a\sum_{n=0}^{\infty}\exp(-a)\frac{a^n}{n!}=a\exp(-a)\exp(a)=a$$

Variance

$$\left\langle X^2 \right\rangle = \sum_{k=0}^{\infty} k^2 \exp(-a) \frac{a^k}{k!}$$

$$= \sum_{k=1}^{\infty} k^2 \exp(-a) \frac{a^k}{k!} = \sum_{k=0}^{\infty} k^2 \exp(-a) a^2 \frac{a^{k-2}}{k(k-1)(k-2)!}$$

$$= a^2 + a$$

$$\sigma_X^2 = a^2 + a - a^2 = a$$

Characteristic function

$$\varphi_X(\omega) = \left\langle \exp\left(i\omega X\right)\right\rangle$$

$$= \sum_{k=0}^{\infty} \exp(i\omega k)\exp(-a)\frac{a^k}{k!}$$

$$= \sum_{k=0}^{\infty} \exp(-a)\frac{\left(a\exp\left(i\omega\right)\right)^k}{k!}$$

$$= \exp\left[a\left\{\exp\left(i\omega\right)-1\right\}\right]$$

Gaussian random variable

$$N(m,\sigma) \Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]; -\infty < x < \infty$$

Area under the curve (=1?)

$$\int_{-\infty}^{\infty} p_X(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx$$

Substitute $u = \left(\frac{x-m}{\sigma}\right) \Rightarrow \sigma du = dx$

$$\Rightarrow \int_{-\infty}^{\infty} p_X(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}u^2\right] du$$

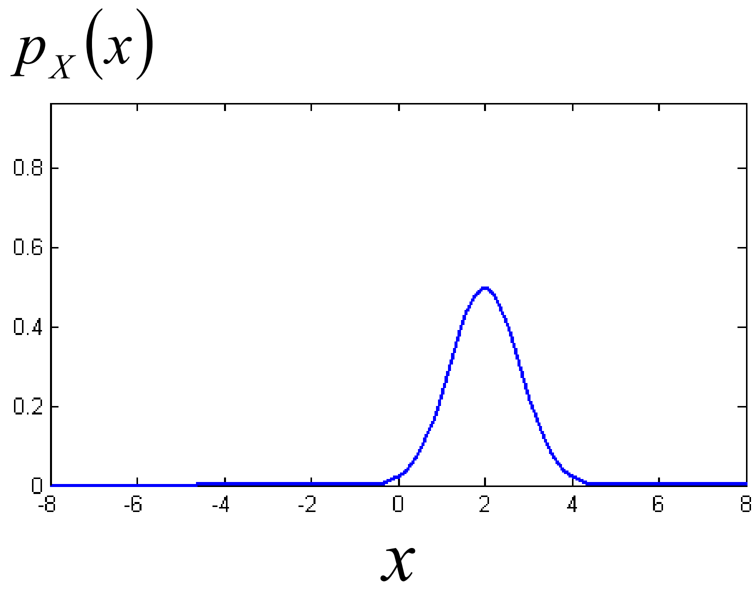
Let I= $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}u^2\right] du$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{u^2+v^2}{2}\right)\right] dudv$$

Substitute $u = r\cos\theta; v = r\sin\theta \Rightarrow rdrd\theta = dudv$

$$\Rightarrow I^2 = \int_0^{\infty}\int_0^{2\pi} r\exp\left[-\frac{r^2}{2}\right] drd\theta = 2\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} p_X(x)\,dx = 1$$



Exercise

Show that

$$\langle X \rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]dx = m$$

$$\left\langle (X - m)^2 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-m)^2 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]dx = \sigma^2$$

$$\left\langle (X - m)^3 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-m)^3 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]dx = 0$$

$$\left\langle (X - m)^4 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-m)^4 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]dx =$$

Skewness=0

Kurtosis=3

$$\phi_X(\omega) = \exp\left(im\omega - \frac{1}{2}\sigma^2\omega^2\right)$$

Remark

A Gaussian random variable is completely specified in terms of its mean and standard deviation.

More on Gaussian random variable

Let $X \sim N(m,\sigma)$.

$$\Rightarrow U = \frac{X - m}{\sigma} \sim N(0,1).$$

$$\Rightarrow p_U(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{u^2}{2}\right] du$$

$$\Rightarrow \langle U \rangle = 0 \;\&\; \langle U^2 \rangle = 1.$$

$$P_U(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left[-\frac{s^2}{2}\right] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left[-\frac{s^2}{2}\right] ds + \frac{1}{\sqrt{2\pi}} \int_{0}^{u} \exp\left[-\frac{s^2}{2}\right] ds$$

$$= 0.5 + \mathrm{erf}(u)$$

$$\mathrm{erf(u)} = \frac{1}{\sqrt{2\pi}} \int_{0}^{u} \exp\left[-\frac{s^2}{2}\right] ds$$

$X \sim N(m,\sigma)$

$\Rightarrow$

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}$$

with $-\infty < x < \infty$

$$U = \frac{X-m}{\sigma}$$

$\Rightarrow -\infty < u < \infty$

$x = \sigma u + m$

$$\frac{du}{dx} = \frac{1}{\sigma}$$

$$p_U(u) = \sigma p_X(x = \sigma u + m)$$

$$= \sigma \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}u^2\right\} \sim N(0,1)$$

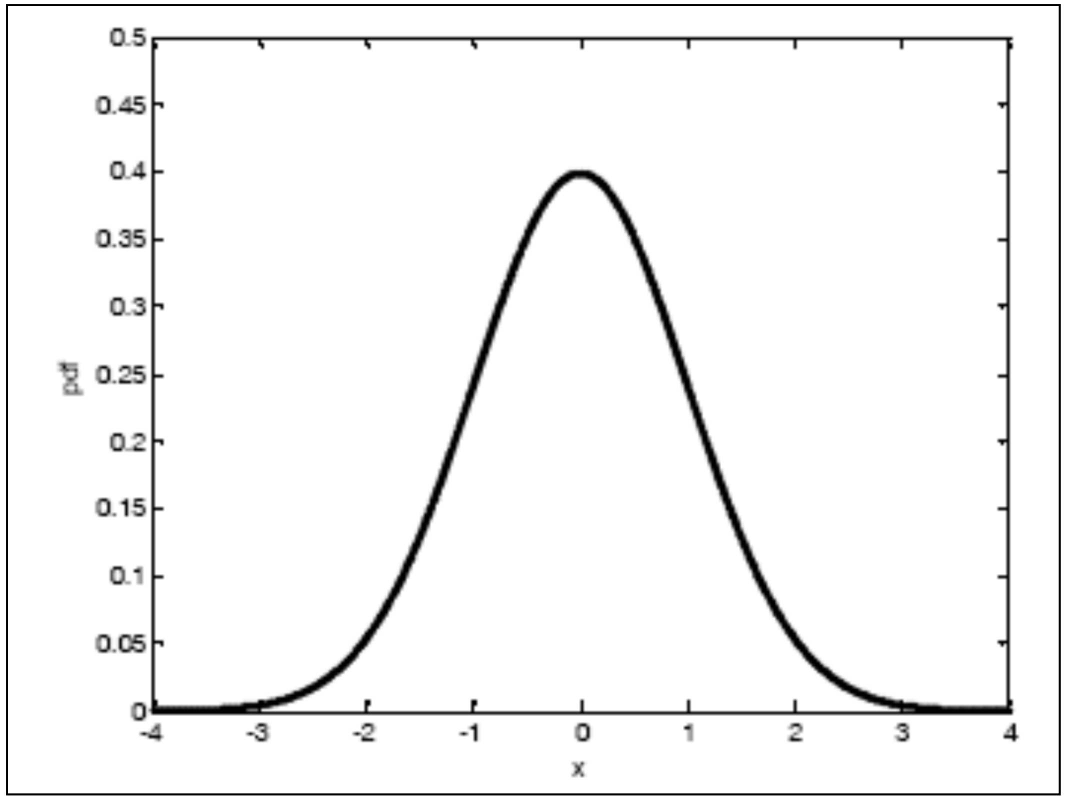
$$\int_{-1}^{1} p_U(u)du = 0.68268$$

$$\int_{-2}^{2} p_U(u)du = 0.95450$$

$$\int_{-3}^{3} p_U(u)du = 0.99730$$

$$\Rightarrow P(m - 3\sigma < X \le m + 3\sigma) = 0.99730$$

$m \pm 3\sigma$ ~ extremes



Moment Genrating function

Let $Z \sim N(0,1)$.

$$\langle \exp(sX) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(sx) \exp\left(-\frac{x^2}{2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 - 2sx}{2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-s)^2}{2} + \frac{s^2}{2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{s^2}{2} \right) \int_{-\infty}^{\infty} \exp\left(-\frac{(x-s)^2}{2} \right) dx$$

$$= \exp\left(\frac{s^2}{2} \right)$$

Moment Genrating function

Let X $\sim N(\mu, \sigma)$.

$$\langle \exp(sX) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(sx) \exp\left(-\left(\frac{x-\mu}{\sigma} \right)^2 \right) dx$$

Substitute $\frac{x-\mu}{\sigma} = u$ & proceed.

Show that $\langle \exp(sX) \rangle = \exp\left(s\mu + \frac{s^2\sigma^2}{2} \right)$

A word of caution.

Moments may not always exist.

Example : Cauchy random variable

$$p_X(x) = \frac{1/\pi}{1+x^2} \quad -\infty < x < \infty.$$

$$\int_{-\infty}^{\infty} \frac{1/\pi}{1+x^2} dx = 1$$

$$\lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} x^n \frac{1/\pi}{1+x^2} dx \to \infty; n = 1, 2, \cdots$$

Mean and standard deviation can be used to obtain bounds on probabilities.

Markov inequality

Let X be a random variable such that it takes non-negative values. That is, $P(X<0)=0$.

Then, for any $a \geq 0$,

$$P(X \geq a) \leq \frac{\langle X \rangle}{a}.$$

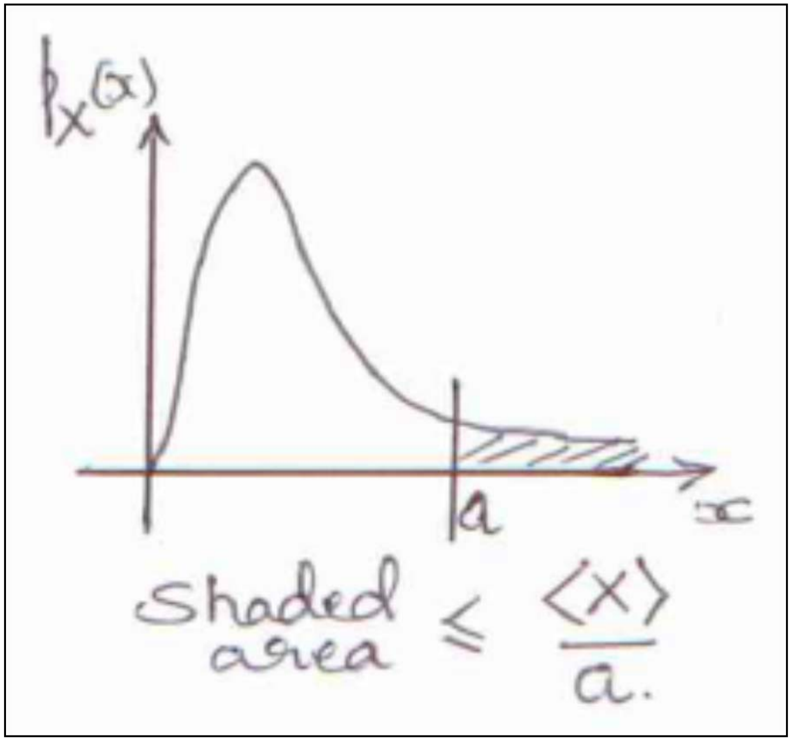
Proof

$$\langle X \rangle = \int_0^{\infty} x p_X(x) dx$$

$$= \int_0^{a} x p_X(x) dx + \int_a^{\infty} x p_X(x) dx$$

$$\geq \int_a^{\infty} x p_X(x) dx \geq \int_a^{\infty} a p_X(x) dx = aP(X \geq a)$$

$$\Rightarrow P(X \geq a) \leq \frac{\langle X \rangle}{a}$$



Chebychev inequality

Let X be a RV with mean μ and standard deviation σ.

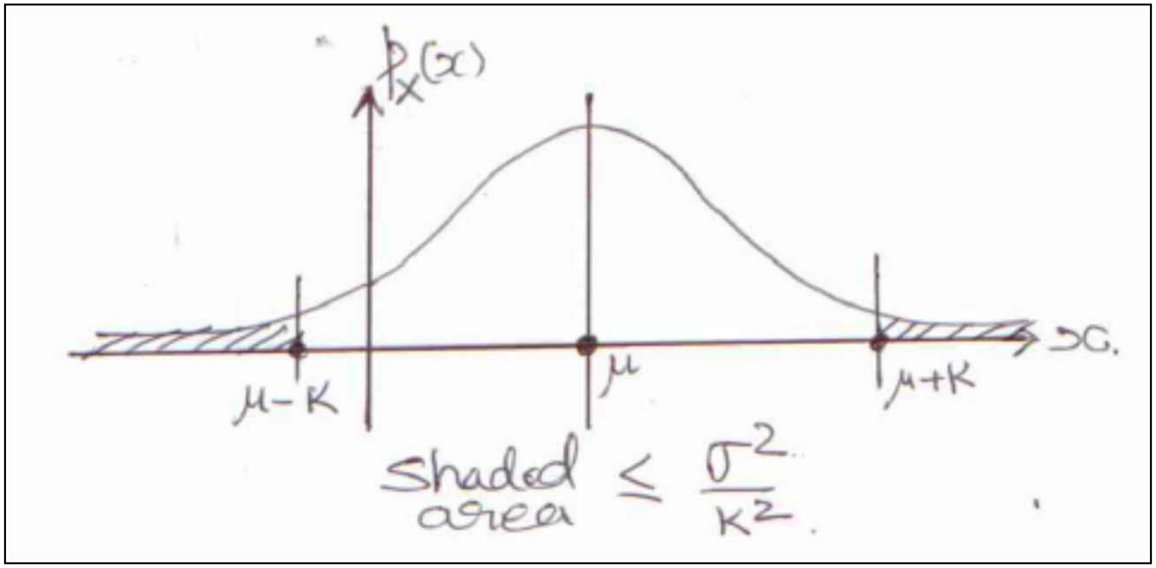
Then $P(|X - \mu| \geq k) \leq \dfrac{\sigma^2}{k^2}$

Proof : Consider $(X - \mu)^2$. This is non-negative. By Markov inequality,

$$P\left((X-\mu)^2 \geq k^2\right) \leq \frac{\left\langle (X-\mu)^2 \right\rangle}{k^2}.$$

$$\left\{(X-\mu)^2 \geq k^2\right\} \Rightarrow \left(|X-\mu| \geq k\right).$$

$$\Rightarrow P\left(|X-\mu| \geq k\right) \leq \frac{\sigma^2}{k^2}.$$



Remarks

(1) The inequalities are exact in nature and are valid for any pdf.

(2) Let σ=0.

$\Rightarrow \mathrm{P}\left(\left|\mathrm{X}\text{-}\mu\right| \geq k\right)=0$

$\Rightarrow P\left(\mathrm{X} \text{ lies outside } (\mu\text{-k},\mu\text{+k})\right)=0.$

Consider the limit of $\mathrm{k} \rightarrow 0$.

$\Rightarrow \mathrm{P}\left[\mathrm{X}=\mu\right]=1.$

$\Rightarrow X$ is deterministic.

(3) We have $\left\langle \mathrm{X}^2 \right\rangle = \sigma_X^2 + m_X^2$.

$\Rightarrow$ If $\left\langle \mathrm{X}^2 \right\rangle = 0 \Rightarrow \sigma_X^2 + m_X^2 = 0 \Rightarrow \sigma_X^2 = 0 \;\&\; m_X^2 = 0.$

$\Rightarrow P\left(X=0\right)=1.$

(4) Chebychev's bound need not provide sharp bounds, that is, the utility as a bound may be limited.

Example:

Let X~N(0,1). Consider $\mathrm{P}\left(\left|\mathrm{X}\right| \geq 3\right)$.

The true value is 0.0027.

Chebychev bound provides $\mathrm{P}\left(\left|\mathrm{X}\right| \geq 3\right) \leq \frac{1}{9}$.

This is not a good enough bound for applications.

(5)If $\frac{\sigma}{\mathrm{k}} > 1,$ the bounds have no significance.

$$0 \leq \mathrm{P}\left(\left|\mathrm{X}\text{-}\mu\right| \geq k\right) \leq \min\left(1, \frac{\sigma_X^2}{k^2}\right).$$

Multi - dimensional random variables

Consider two random variables X and Y.

Define $E_1 = \{X \le x\}$ and $E_2 = \{Y \le y\}$.

$E = E_1 \cap E_2 = \{X \le x \cap Y \le y\}$.

Definition

$$P_{XY}(x, y) = P\{X \le x \cap Y \le y\} = P\{X \le x, Y \le y\}$$

Note: Comma (,) denotes intersection ($\cap$).

$P_{XY}(x, y)$ = Joint probability distribution function of X and Y (JPDF)

Definition

$$p_{XY}(x, y) = \frac{\partial^2 P_{XY}(x, y)}{\partial x \partial y}$$

$p_{XY}(x, y)$ = Joint probability density function of X and Y (jpdf).

Remarks

(1) Geometric interpretation : Place a point randomly in the x - y plane.

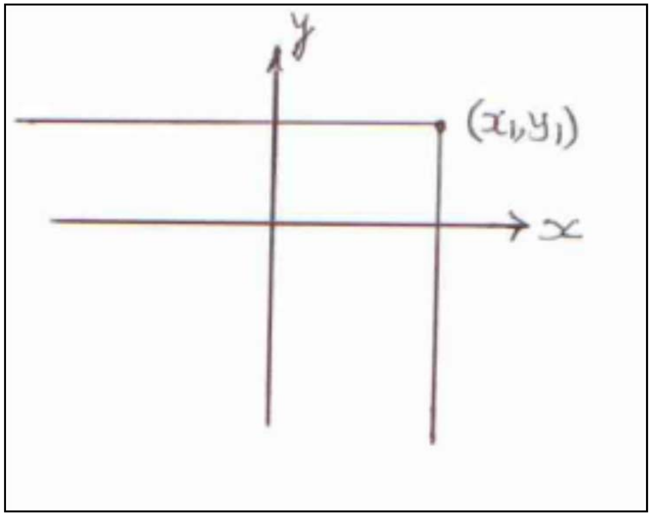
$P_{XY}(x_1, y_1) = P(\text{The point lies in the region } X \le x_1 \cap Y \le y_1).$

(2) $P_{XY}(x,\infty) = P(X \le x \cap Y \le \infty) = P(X \le x) = P_X(x).$

(3) $P_{XY}(\infty, y) = P(X \le \infty \cap Y \le y) = P(Y \le y) = P_Y(y).$

(4) $P_{XY}(\infty,\infty) = P(X \le \infty \cap Y \le \infty) = P(\Omega) = 1.$

(5) $P_{XY}(x,-\infty) = P_{XY}(-\infty, y) = 0.$



Remarks (Continued)

(6) $P\left(X \le x_1 \cap y_1 < Y \le y_2\right) = ?$

Define

$S_1 = \left(X \le x_1 \cap Y \le y_2\right)$

$S_2 = \left(X \le x_1 \cap Y \le y_1\right)$

$S_3 = \left(X \le x_1 \cap y_1 < Y \le y_2\right)$

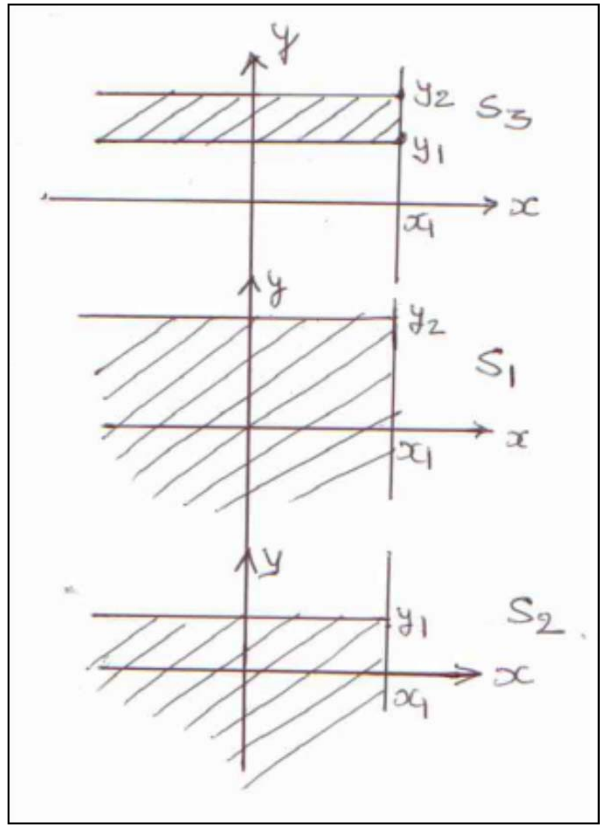
$S_1 = S_2 \cup S_3; S_2 \cap S_3 = \varphi$

$P(S_1) = P\left(S_2\right) + P\left(S_3\right)$

$P_{XY}\left(x_1, y_2\right) = P_{XY}\left(x_1, y_1\right) + P\left(X \le x_1 \cap y_1 < Y \le y_2\right)$

$\Rightarrow P\left(X \le x_1 \cap y_1 < Y \le y_2\right) = P_{XY}\left(x_1, y_2\right) - P_{XY}\left(x_1, y_1\right)$

(7) $P\left(x_1 < X \le x_2 \cap y_1 < Y \le y_2\right) = ?$ (Exercise)



Remarks (Continued)

(8) $p_{XY}(x,y) = \dfrac{\partial^2 P_{XY}(x,y)}{\partial x \partial y}$

$\Rightarrow P_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} p_{XY}(u,v)\,du\,dv$

$\Rightarrow P_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}(u,v)\,du\,dv = P(\Omega) = 1$

(9) $P_{XY}(\infty,y) = P_Y(y)$

$\Rightarrow P_Y(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} p_{XY}(u,v)\,du\,dv$

$\Rightarrow p_Y(y) = \dfrac{dP_Y(y)}{dy} = \int_{-\infty}^{\infty} p_{XY}(x,y)\,dx$

Similarly, $p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)\,dy$

$p_X(x) =$ Marginal pdf of X

$p_Y(y) =$ Marginal pdf of Y

(10) Knowing $p_{XY}(x,y)$ we can find the marginal pdf-s. The other way is not true.

(11) $P_{XY}(x,y)$ is monotone nondecreasing in x and y.

$\Rightarrow p_{XY}(x,y) \geq 0.$

Complete specification of two random variables *X* and *Y* through JPDF or jpdf.

JCSS (2002)

Steel as a 5-dimensional random variable

Description	COV
Yield strength	0.07
Ultimate tensile strength	0.04
Young's modulus	0.03
Poisson's ratio	0.03
Ultimate strain	0.06

$$\rho = \begin{bmatrix} 1 & 0.75 & 0 & 0 & -0.45 \\ & 1 & 0 & 0 & -0.60 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

Distribution: Multivariate lognormal random variable

Independence of random variables

Recall

$$P(A|B)=\frac{P(A\cap B)}{P(B)};P(B)\neq 0.$$

$$A\perp B\Rightarrow P(A\cap B)=P(A)P(B)$$

Define $A=\{X\leq x\}$ and $B=\{Y\leq y\}$

$$X\perp Y\Rightarrow P(X\leq x\cap Y\leq y)=P(X\leq x)P(Y\leq y)$$

$$\Rightarrow P_{XY}(x,y)=P_X(x)P_Y(y)$$

$$\Rightarrow p_{XY}(x,y)=p_X(x)p_Y(y)$$

Remark

If X and Y are independent, the complete specification of X and Y will be through $p_X(x)$ & $p_Y(y)$.

Conditional distributions

Let X and Y be two random variables.

Let B be an event.

Define

$$P\left(Y \le y \mid B\right) = \frac{P\left(Y \le y \cap B\right)}{P\left(B\right)} = P_Y\left(y \mid B\right)$$

$P_Y\left(y \mid B\right) =$ Conditional PDF of Y given B.

Similarly, we introduce $p_Y\left(y \mid B\right) = \dfrac{\partial P_Y\left(y \mid B\right)}{\partial y}$

$p_Y\left(y \mid B\right) =$ conditional pdf of Y given B.

Remarks

(1) $P_Y(y \mid B)$ has all the properties of a PDF.

Similarly, $p_Y(y|B)$ will have all the properties of a pdf.

(2) One can define conditional expectation

$$\langle g(Y)|B\rangle = \int_{-\infty}^{\infty} g(y)\, p_Y(y|B)\, dy$$

Thus we could define conditional mean, conditional variance, etc.

Remarks (Continued)

(3) Let $B = \{X \le x\}$.

$$\Rightarrow P(Y \le y \mid X \le x) = \frac{P(Y \le y \cap X \le x)}{P(X \le x)} = \frac{P_{XY}(x, y)}{P_X(x)}$$

$$\Rightarrow P_Y(y \mid X \le x) = \frac{P_{XY}(x, y)}{P_X(x)} = \frac{\int_{-\infty}^{x}\int_{-\infty}^{y} p_{XY}(u, v)\,du\,dv}{\int_{-\infty}^{x}\int_{-\infty}^{\infty} p_{XY}(u, v)\,du\,dv}$$

$$\Rightarrow p_Y(y \mid X \le x) = \frac{dP_Y(y \mid X \le x)}{dy} = \frac{\int_{-\infty}^{x} p_{XY}(u, y)\,du}{\int_{-\infty}^{x}\int_{-\infty}^{\infty} p_{XY}(u, v)\,du\,dv}$$

Remarks (Continued)

(4)Let $B=\{x_1<X\le x_2\}$

$$\Rightarrow P_Y(y\mid B)=\frac{P(Y\le y\cap x_1<X\le x_2)}{P(x_1<X\le x_2)}$$

$$=\frac{P_{XY}(x_2,y)-P_{XY}(x_1,y)}{P_X(x_2)-P_X(x_1)}$$

$$=\frac{\int_{-\infty}^{x_2}\int_{-\infty}^{y}p_{XY}(u,v)\,dudv-\int_{-\infty}^{x_1}\int_{-\infty}^{y}p_{XY}(u,v)\,dudv}{\int_{x_1}^{x_2}p_X(x)\,dx}$$

$$\Rightarrow p_Y(y\mid B)=\frac{dP_Y(y\mid B)}{dy}$$

$$=\frac{\int_{-\infty}^{x_2}p_{XY}(u,y)\,du-\int_{-\infty}^{x_1}p_{XY}(u,y)\,du}{\int_{x_1}^{x_2}p_X(x)\,dx}$$

$$\Rightarrow p_Y(y\mid B)=\frac{\int_{x_1}^{x_2}p_{XY}(u,y)\,du}{\int_{x_1}^{x_2}p_X(x)\,dx}$$

Remarks (Continued)

(5) Let $x_1=x$ and $x_2=x+dx$

$$p(y\mid B)=\frac{p_{XY}(x,y)dx}{p_X(x)dx}=\frac{p_{XY}(x,y)}{p_X(x)}$$

As $dx\to 0,\ B=\{X=x\}$.

Accordingly, one gets

$$p(y|X=x)=\frac{p_{XY}(x,y)}{p_X(x)}$$

Joint Expectations :

Let X and Y be two random variables.

Consider a function $g(X,Y)$.

Definition

$$\langle g(X,Y)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)p_{XY}(x,y)dxdy$$

Remarks

$$(1)\; m_{nk} = \left\langle X^n Y^k \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k p_{XY}(x,y)dxdy$$

$$\text{Clearly, } \mathrm{m}_{10} = \langle X \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x p_{XY}(x,y)dxdy$$

$$= \int_{-\infty}^{\infty} x p_X(x)dx = \eta_X.\ \text{Similarly, } \mathrm{m}_{01} = \eta_Y.$$

$$(2)\, \mu_{\mathrm{nk}} = \left\langle (X-\eta_X)^n (Y-\eta_Y)^k \right\rangle.$$

Clearly, $\mu_{20} = \sigma_X^2$ & $\mu_{02} = \sigma_Y^2$.

(3) **Definition**

$$\mu_{11} = \langle (X-\eta_X)(Y-\eta_Y) \rangle = \sigma_{XY}$$

σ_{XY} is the covariance of RVs X and Y.

(4) **Definition**

$$r_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \text{correlation coefficient between } X \text{ and } Y.$$

More on Covariance and Correlation Coefficient

(1) Let $Y = aX + b$

$$\langle Y \rangle = a\langle X \rangle + b = a\eta_X + b$$

$$\sigma_Y^2 = \left\langle (Y - \eta_Y)^2 \right\rangle = \left\langle (aX + b - a\eta_X - b)^2 \right\rangle = a^2\sigma_X^2$$

$$\sigma_{XY} = \langle (X - \eta_X)(Y - \eta_Y) \rangle$$

$$= \langle (X - \eta_X)(aX + b - a\eta_X - b) \rangle$$

$$= a\langle (X - \eta_X)(X - \eta_X) \rangle = a\sigma_X^2$$

$$r_{XY} = \frac{a\sigma_X^2}{\sigma_X \sqrt{a^2\sigma_X^2}} = \pm 1 \text{ (sign of } a\text{)}$$

(2) Let $X \perp Y$ $\left[\Rightarrow p_{XY}(x, y) = p_X(x)p_Y(y)\right]$

$$\sigma_{XY} = \langle (X - \eta_X)(Y - \eta_Y) \rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \eta_X)(y - \eta_Y) p_{XY}(x, y)\, dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \eta_X)(y - \eta_Y) p_X(x) p_Y(y)\, dx dy = 0$$

$$\Rightarrow r_{XY} = 0$$

More on Covariance and Correlation Coefficient

(3) **Bounds on** r_{XY}

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\eta_X)(y-\eta_Y)\,p_{XY}(x,y)\,dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\eta_X)\sqrt{p_{XY}(x,y)}\,(y-\eta_Y)\sqrt{p_{XY}(x,y)}\,dxdy$$

$$\le \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\eta_X)^2\,p_{XY}(x,y)\,dxdy\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y-\eta_Y)^2\,p_{XY}(x,y)\,dxdy\right]^{\frac{1}{2}}$$

(by virtue of the Schwarz inequality)

$$= \pm\sigma_X\sigma_Y$$

$$\Rightarrow -1 \le \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = r_{XY} \le 1$$

(4)**Definition**

$r_{XY} = 0 \Rightarrow X$ and Y are uncorrelated.

(5) $X \perp Y \Rightarrow X$ and Y are uncorrelated

X and Y are uncorrelated does not mean that $\mathrm{X} \perp \mathrm{Y}$

Summary

$r_{XY} = \pm 1$ are the limits of linear behavior;

$r_{XY} = 0 \Rightarrow X$ and Y are uncorrelated.

2 - dimensional Gaussian random variable

X and Y are said to be jointly Gaussian if

$$p_{XY}(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{\left(1-r_{XY}^2\right)}}\exp\left[-\frac{1}{2\left(1-r_{XY}^2\right)}\left\{\left(\frac{x-\eta_X}{\sigma_X}\right)^2+\left(\frac{y-\eta_Y}{\sigma_Y}\right)^2-\frac{2r_{XY}(x-\eta_X)(y-\eta_Y)}{\sigma_X\sigma_Y}\right\}\right]$$

$-\infty<x<\infty;-\infty<y<\infty$

Exercise: Prove that

(a) $\langle X\rangle=\eta_X;\langle Y\rangle=\eta_Y;\left\langle (X-\eta_X)^2\right\rangle=\sigma_X^2;\left\langle (Y-\eta_Y)^2\right\rangle=\sigma_Y^2;\left\langle (X-\eta_X)(Y-\eta_Y)\right\rangle=r_{XY}\sigma_X\sigma_Y$

Notes: $\begin{pmatrix}X\\Y\end{pmatrix}\sim N\left[\begin{pmatrix}\eta_X\\\eta_Y\end{pmatrix}\ \begin{pmatrix}\sigma_X^2 & r_{XY}\sigma_X\sigma_Y\\ r_{XY}\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}\right]$

$\begin{pmatrix}\sigma_X^2 & r_{XY}\sigma_X\sigma_Y\\ r_{XY}\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}$ is known as the covariance matrix.

(b) Show that $p_Y(y\,|\,X=x)=\dfrac{p_{XY}(x,y)}{p_X(x)}=\dfrac{1}{\sigma_Y\sqrt{2\pi\left(1-r_{XY}^2\right)}}\exp\left[-\dfrac{\left(y-r_{XY}\sigma_Y x/\sigma_X\right)^2}{2\sigma_Y^2\left(1-r_{XY}^2\right)}\right];-\infty<y<\infty$

Remarks

(a) $r_{XY} = 0 \Rightarrow p_{XY}(x, y) = p_X(x)\, p_Y(y)$

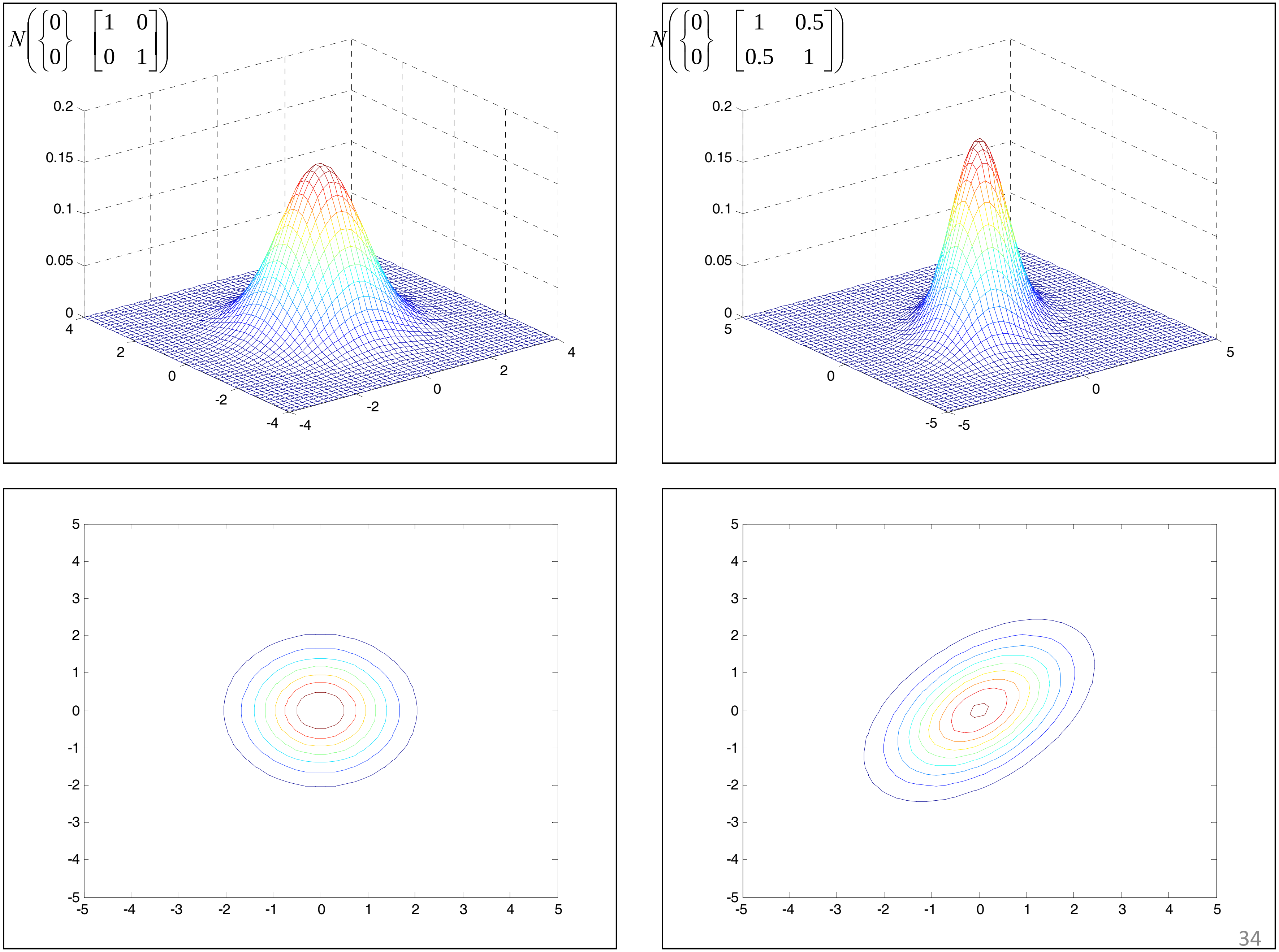
That is, for Gaussian random variables,

$r_{XY} = 0 \Leftrightarrow \mathrm{X} \perp \mathrm{Y}$

(b) Exercise: For $r_{XY} \neq 0,$ prove that

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x, y)\, dy = \frac{1}{\sqrt{2\pi\sigma_X}} \exp\left(-\frac{1}{2}\left(\frac{x - \eta_X}{\sigma_X} \right)^2 \right); -\infty < x < \infty$$

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x, y)\, dx = \frac{1}{\sqrt{2\pi\sigma_Y}} \exp\left(-\frac{1}{2}\left(\frac{y - \eta_Y}{\sigma_Y} \right)^2 \right); -\infty < y < \infty$$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4
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5
-3
-1
1
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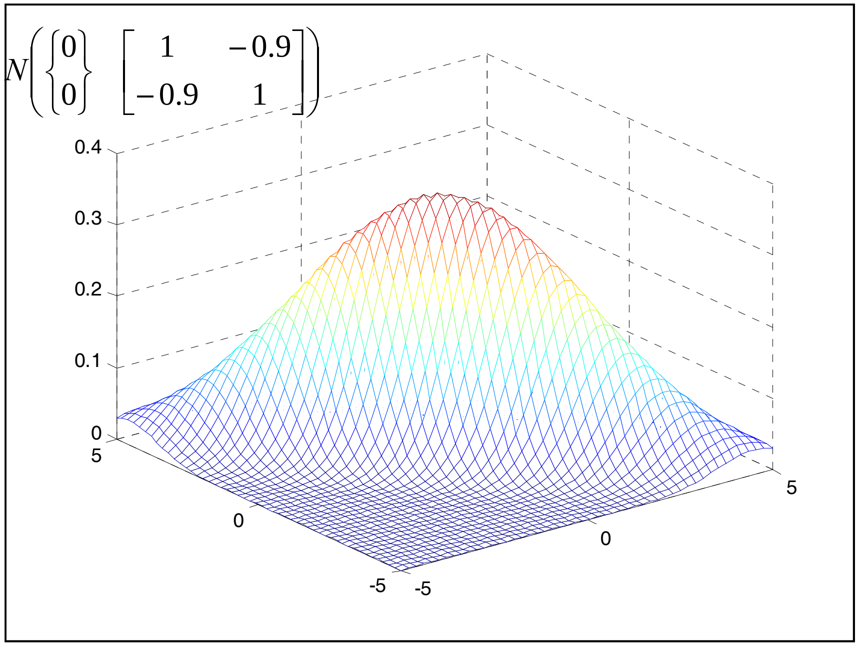
$N\left(\begin{Bmatrix}0\\0\end{Bmatrix}\begin{bmatrix}1 & -0.9\\-0.9 & 1\end{bmatrix}\right)$
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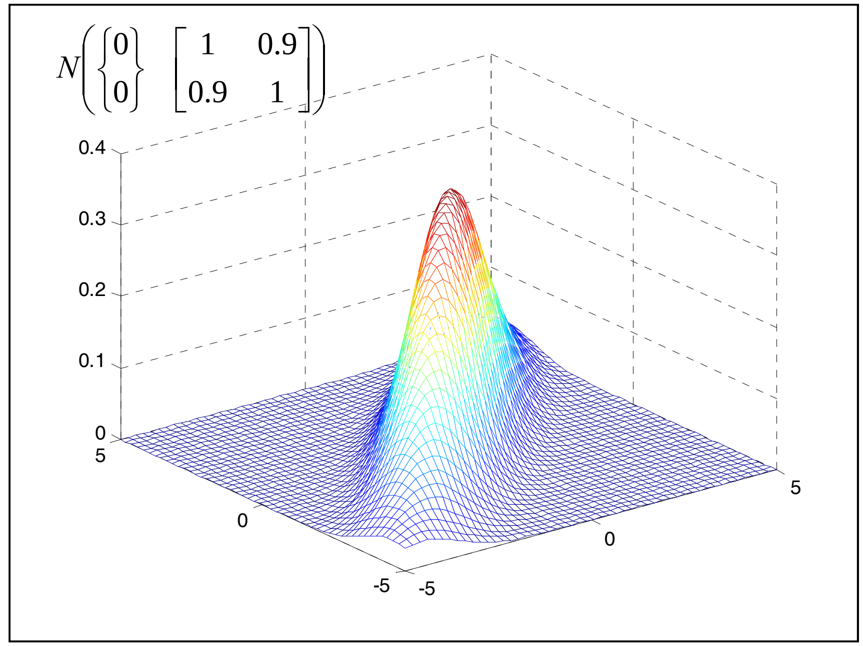
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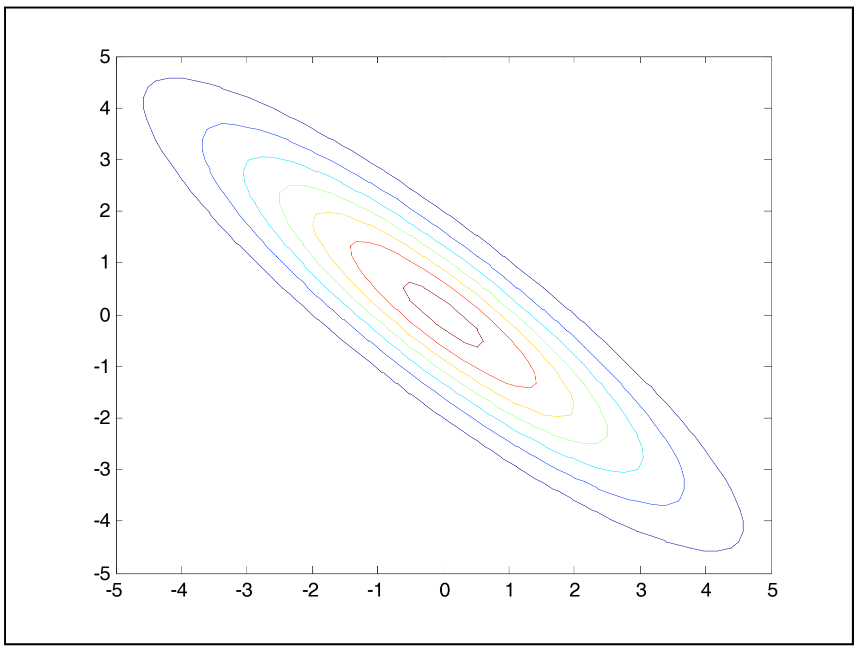
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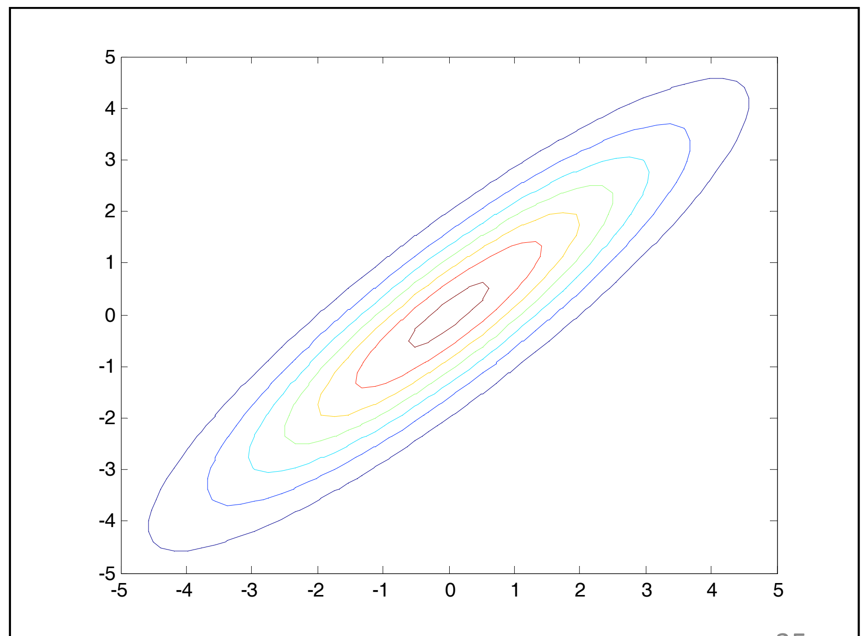
5 4 3 2 1 0 -1 -2 -3 -4 -5
-5 -4 -3 -2 -1 0 1 2 3 4 5

Functions of random variables

Let X and Y be two random variables. Define $U = g(X,Y)$ and $V = h(X,Y)$.

Question: Given the jpdf of X and Y, what is the jpdf of U and V?

Steps

(1) Consider $u = g(x,y)$ and $v = h(x,y)$.

Solve for (x,y) from these equaitons.

Let $\left(x_i, y_i\right)_{i=1}^{n}$ be the roots. Note that n could be ∞.

(2) Determine

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ OR } J^{-1} = \frac{1}{J} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Steps (continued)

(3)Find

$$p_{UV}(u,v) = \sum_{i=1}^{n} \left. \frac{p_{XY}(x_i,y_i)}{|J|} \right|_{\substack{x=x_i \\ y=y_i}}$$

OR $p_{UV}(u,v) = \sum_{i=1}^{n} \left| J^{-1} \right| p_{XY}(x_i,y_i) \Big|_{\substack{x=x_i \\ y=y_i}}$

(4)Examine g and h and decide upon limits of u and v.

Note : Decide if it is easier to work with J or J^{-1}

Example − 1

$$U = X + Y \qquad V = X - Y$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$u = x + y$$

$$v = x - y$$

$$x = \frac{u+v}{2}; y = \frac{u-v}{2}$$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2; \quad |J| = 2.$$

$$p_{UV}(u,v) = \left. \frac{p_{XY}(x,y)}{|J|} \right|_{\substack{x = \frac{u+v}{2} \\ y = \frac{u-v}{2}}}$$

$$= \frac{1}{2\pi}\frac{1}{2}\exp\left[-\frac{1}{2}\left(\frac{u+v}{2}\right)^2 - \frac{1}{2}\left(\frac{u-v}{2}\right)^2 \right]$$

$$= \frac{1}{4\pi}\exp\left[-\frac{1}{4}\left(u^2 + v^2\right) \right]; -\infty < u < \infty; -\infty < v < \infty$$

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N\left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \right)$$

Check

$$\langle U \rangle = \langle X \rangle + \langle Y \rangle = 0 \text{ (ok)}$$

$$\langle V \rangle = \langle X \rangle - \langle Y \rangle = 0 \text{ (ok)}$$

$$\langle \mathrm{U}^2 \rangle = \langle X^2 + Y^2 + 2XY \rangle = 2 = \sigma_U^2 \text{ (ok)}$$

$$\langle \mathrm{V}^2 \rangle = \langle X^2 + Y^2 - 2XY \rangle = 2 = \sigma_V^2 \text{ (ok)}$$

$$\langle \mathrm{U}V \rangle = \langle X^2 - Y^2 \rangle = 0 = \sigma_{UV} \text{ (ok)}$$

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$$p_U(u) = \int_{-\infty}^{\infty} p_{UV}(u,v)dv \sim N\left(0, \sqrt{2}\right)$$

$$p_v(v) = \int_{-\infty}^{\infty} p_{UV}(u,v)du \sim N\left(0, \sqrt{2}\right)$$

It turns out that U and V are independent

$$\because \; p_{UV}(u,v) = p_U(u)p_V(v)$$