Stochastic Structural Dynamics

Lecture-3

Scalar random variables-2

Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

Recall

Random variable is a function from sample space into real line such that

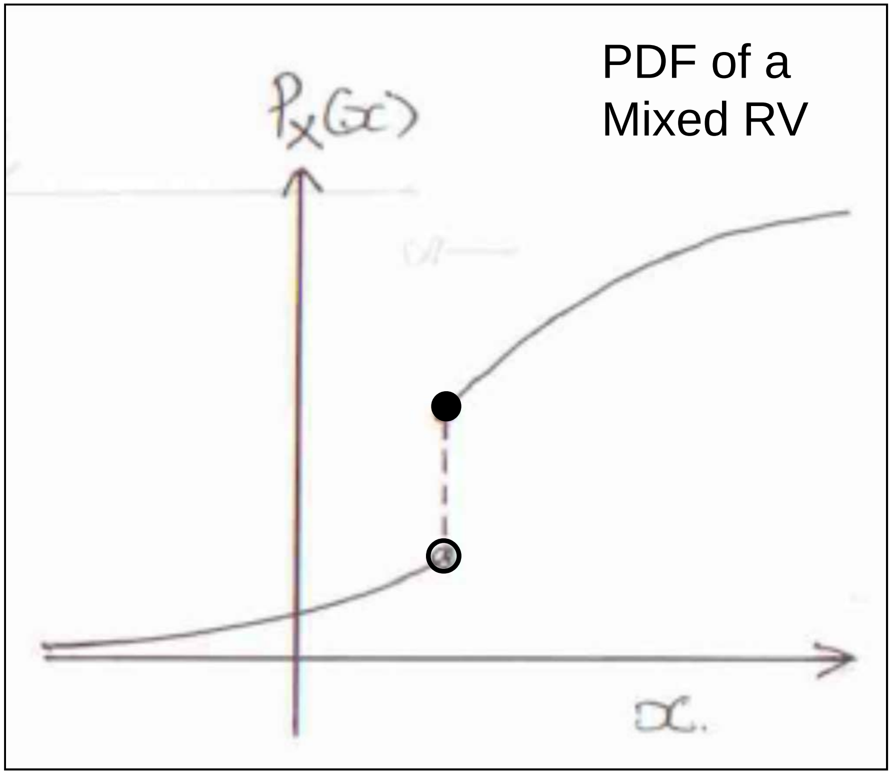
(1) for every $x \in R$, $\{\omega : X(\omega) \le x\}$ is an event,

(2) $P(\omega : X(\omega) = \pm\infty) = 0$

- Discrete
- Continuous
- Mixed

Descriptions

- Probability distribution function
- Probability density function
- Probability mass function



Recall (continued)

Bernoulli's random variable

$P[X=0]=p$

$P[X=1]=1-p$

Binomial random variable

$P[N=k]={}^{N}C_{k}\,p^{k}(1-p)^{N-k}\quad k=0,1,2,\dots N$

Geometric random variable

$P[X=k]=(1-p)^{k-1}p;\,k=1,2,\cdots,\infty$

Models for rare events : Poisson random variable

(a) We are looking for occurrence of an isolated phenomenon in a time/space continuum.

(b) We cannot put an upper bound on the number of occurrences.

(c) Actual number of occurrences is relatively small.

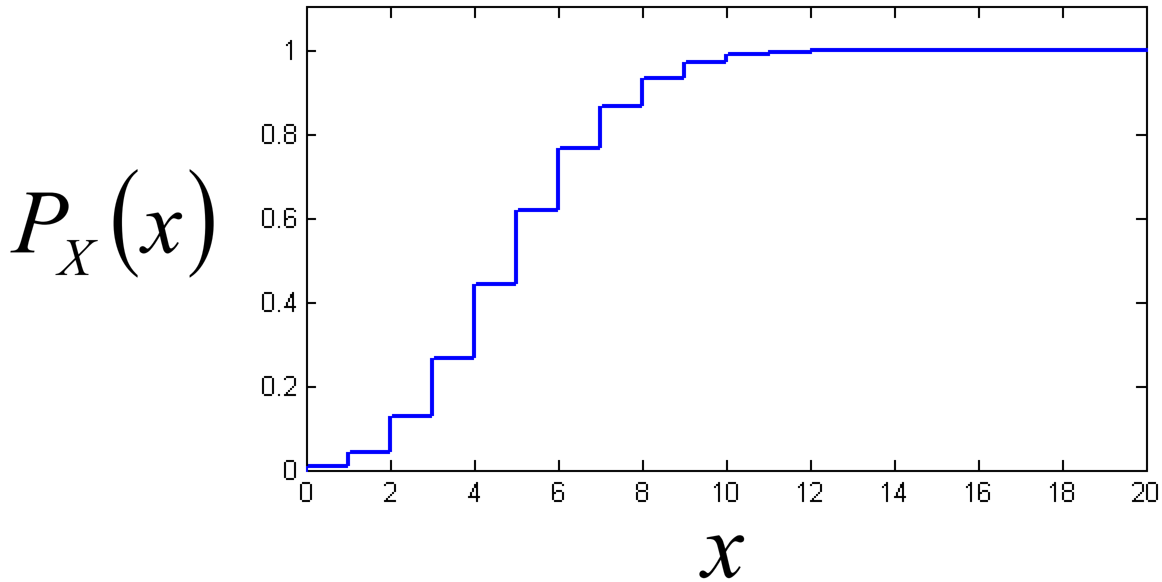
Examples : goals in football match (time continuum), defect in a yarn (1 - d space continuum), typos in a manuscript (2 - d continuum), defect in a solid (3 - d continuum). **Stress at a point exceeding elastic limit during the life time of a structure.**

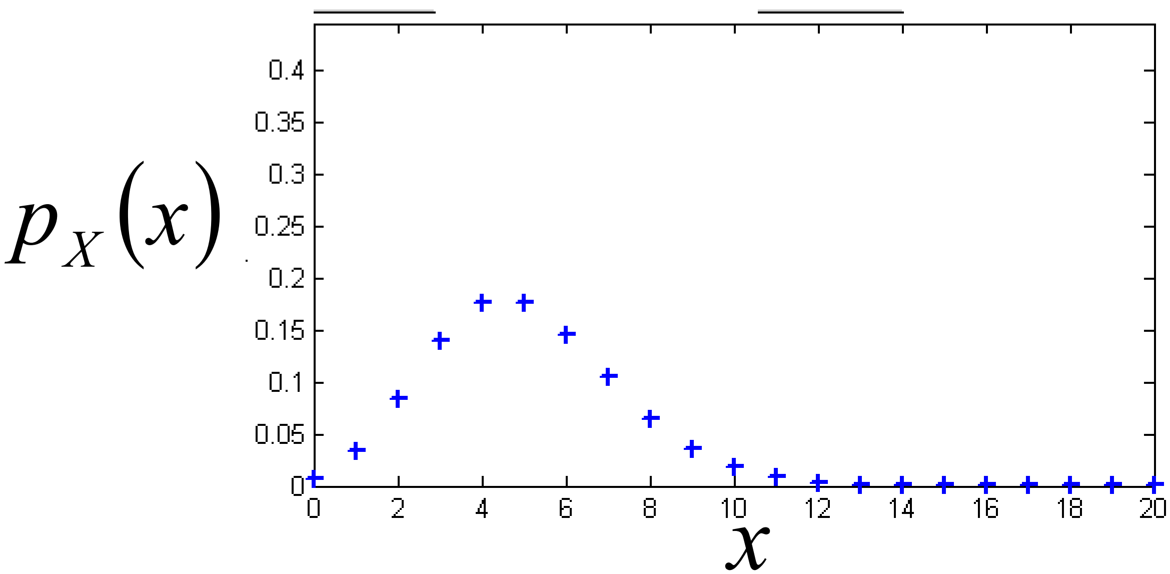
$$P(X=k)=\exp(-a)\frac{a^k}{k!};k=0,1,2,\cdots\infty$$

Check

$$P(X\le\infty)=\sum_{k=0}^{\infty}\exp(-a)\frac{a^k}{k!}=\exp(-a)\sum_{k=0}^{\infty}\frac{a^k}{k!}=\exp(-a)\exp(a)=1.$$

Poisson random variable with a=5





- Discrete RV
- Countably infinite sample space
- Useful in wide variety of contexts

Poisson PDF as a limit of Binomial PDF

The PMF of B(n,p) is given by

$$P\left(X=k\right)={}^{n}C_{k}\,p^{k}(1-p)^{n-k};k=0,1,2,\cdots,n$$

Let $n\rightarrow\infty$, $p\rightarrow 0$, $k<<n$, such that $np\rightarrow a$, ("rare event")

$${}^{n}C_{k}\,p^{k}(1-p)^{n-k}\rightarrow\exp(-a)\frac{a^{k}}{k!}$$

Proof

$$ {}^{n}C_k\, p^k (1-p)^{n-k} = \frac{n!}{(n-k)!k!}\left(\frac{a}{n}\right)^k \left(1-\frac{a}{n}\right)^{n-k} $$

$$ {}^{n}C_k = \frac{n!}{(n-k)!k!} \approx \frac{n^k}{k!} $$

$$ p^k = \left(\frac{a}{n}\right)^k $$

$$ 1-p \approx \exp(-p) $$

$$ (1-p)^{n-k} \approx \exp\left[-\left(n-k\right)p\right] = \exp(-a)\exp(kp) \approx \exp(-a) $$

$$ \Rightarrow {}^{n}C_k\, p^k (1-p)^{n-k} \to \frac{n^k}{k!}\left(\frac{a}{n}\right)^k \exp(-a) = \frac{a^k}{k!}\exp(-a) $$

Example

- Consider $B\left(1000, 10^{-3}\right)$
- $N = 1000;\ p = 10^{-3}$
- $P(X=0) = {}^{1000}C_0 \left(10^{-3}\right)^0 \left(1-10^{-3}\right)^{1000-0} = 0.36769$

Poisson limit

$Np = 1000 \times 10^{-3} = 1$

$k = 0 << 1000$

$$P[X=0] = \exp(-a)\frac{a^0}{0!} = \exp(-1) = 0.36787$$

Example

Consider $B(3000, 10^{-3})$

$P(X > 5) = ?$

$$P(X > 5) = 1 - P(X \le 5) = \sum_{k=0}^{5} {}^{3000}C_k \left(10^{-3}\right)^k \left(1 - 10^{-3}\right)^{3000-k}$$

Try evaluating this!!

Poisson's limit

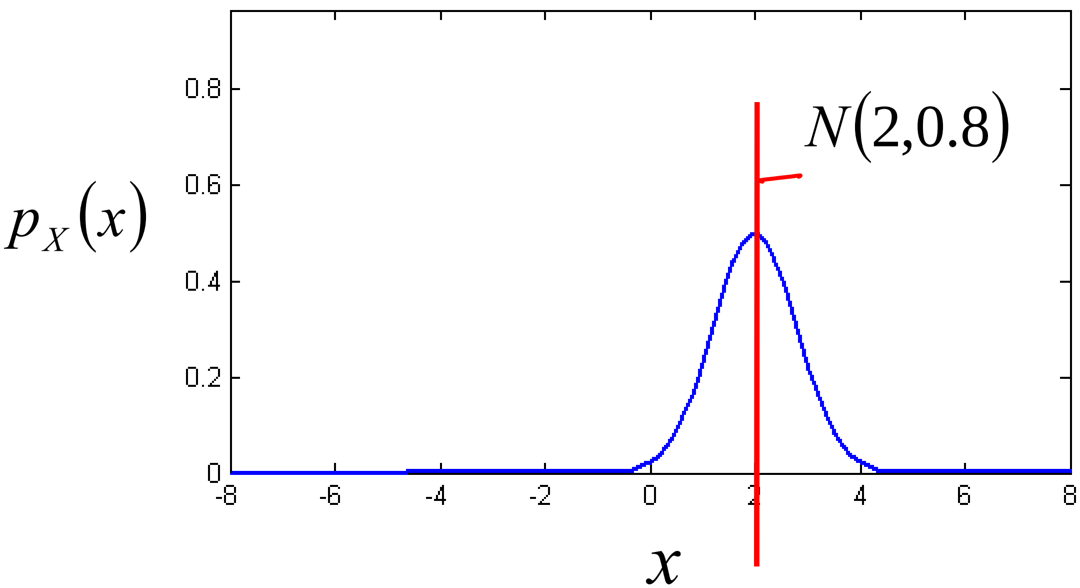
$N = 3000;\ p = 10^{-3};\ a = Np = 3$

$k \le 5 << 3000$

$$P(X > 5) = 1 - \sum_{k=0}^{5} \exp(-3)\frac{3^k}{k!} = 0.084$$

Gaussian (Normal) random variable: model for sums

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m)^2}{\sigma^2} \right); \quad -\infty < x < \infty$$



m and σ are the paramters of a Gaussian random variable.

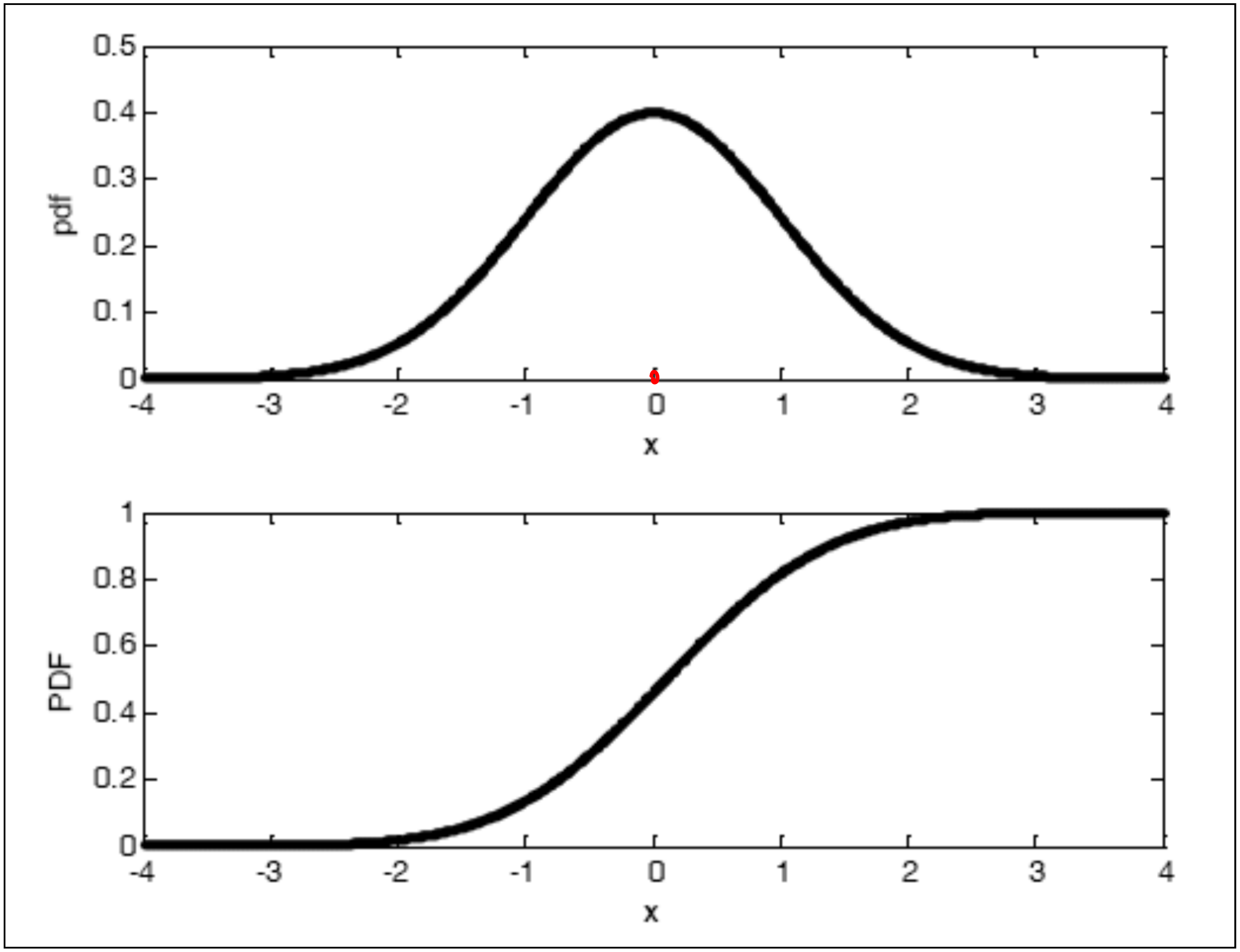
$-\infty < m < \infty$; $\sigma \geq 0$.

Gaussian random variable is denoted by $N(m, \sigma)$

Standard normal random variable

m=0 and σ=1

$$N(0,1)$$



Gaussian random variable as a limit of Binomial random variable

Under the conditions

$$n \to \infty,\ npq >> 1,\ np - \sqrt{npq} < k < np + \sqrt{npq}$$

it can be shown that

$$^{n}C_{k}\, p^{k} (1-p)^{n-k} \to \frac{1}{\sqrt{2\pi npq}} \exp\left[-\frac{1}{2} \left(\frac{k-np}{\sqrt{npq}} \right)^{2} \right]$$

$q = 1 - p$

Example

Let $n = 1000;\ p = 0.5;\ k = 500.$

$$npq = \frac{1000}{4} = 250; \sqrt{npq} = 15.81;$$

$$np - \sqrt{npq} = 500 - 15.81 \approx 485$$

$$np + \sqrt{npq} = 500 + 15.81 \approx 515.$$

It is verified that $485 < k = 500 < 515.$

$$P\left(X = 500\right) = {}^{1000}C_{500}\left(0.5\right)^{500}(1 - 0.5)^{1000-500}$$

$$\approx \frac{1}{\sqrt{2\pi 15.81}} \exp\left[-\frac{1}{2}\left\{\frac{(500-500)^2}{15.81}\right\}\right] = 0.1030$$

Random points :

Poisson models for counting &

exponential models for waiting times

Consider a time interval $0, T$.

Let us place n points randomly is $0, T$.

Let $0 < t_1 < t_2 < T$.

Define $\{\text{success}\} = \{\text{point lies in the sub-interval } t_1, t_2\}$.

$$P\{\text{success}\} = \frac{t_2 - t_1}{T} = p; P\{\text{failure}\} = 1 - p.$$

Define : X = number of points in t_1, t_2.

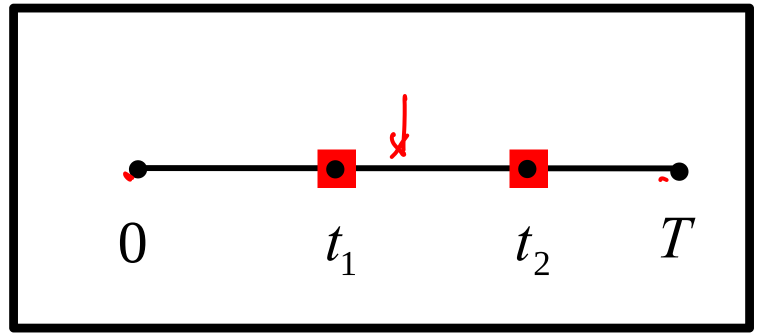
$$P(X = k) = {}^nC_k p^k (1-p)^{n-k}.$$

Now, consider $n \to \infty$, $\dfrac{t_2 - t_1}{T} \to 0, \dfrac{n(t_2 - t_1)}{T} \to a$

$$\Rightarrow {}^nC_k p^k (1-p)^{n-k} \to \exp(-a)\frac{a^k}{k!}.$$

Let $\dfrac{n}{T} = \lambda; (t_2 - t_1) = t_a \Rightarrow a = \lambda t_a$

$$\Rightarrow P(X = k) = \exp(-\lambda t_a)\frac{(\lambda t_a)^k}{k!}; k = 0,1,2,\cdots,\infty$$



Exponetial random variable

Define T^* = time for the first arrival.

$P(T^* > t) = P(\text{no points in } 0,t) = \exp(-\lambda t)$

$\Rightarrow P_{T^*}(t) = 1 - \exp(-\lambda t)$

$\Rightarrow p_{T^*}(t) = \lambda \exp(-\lambda t);\ \lambda > 0.$

Remarks

(a) T^* can be used to model life time of a structure.

(b) $P_{T^*}(t)$ is the generalization of the geometric random variable to the continuous case.

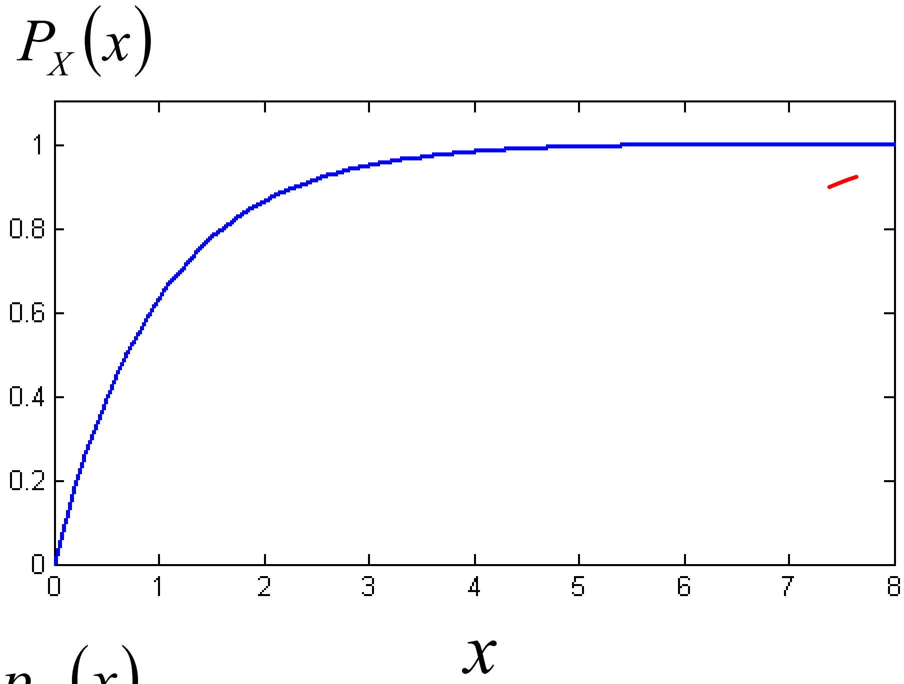
(c) Consider $P(T^* \le t \mid T^* > t_0) = \dfrac{P(T^* \le t \cap T^* > t_0)}{P(T^* > t_0)}$

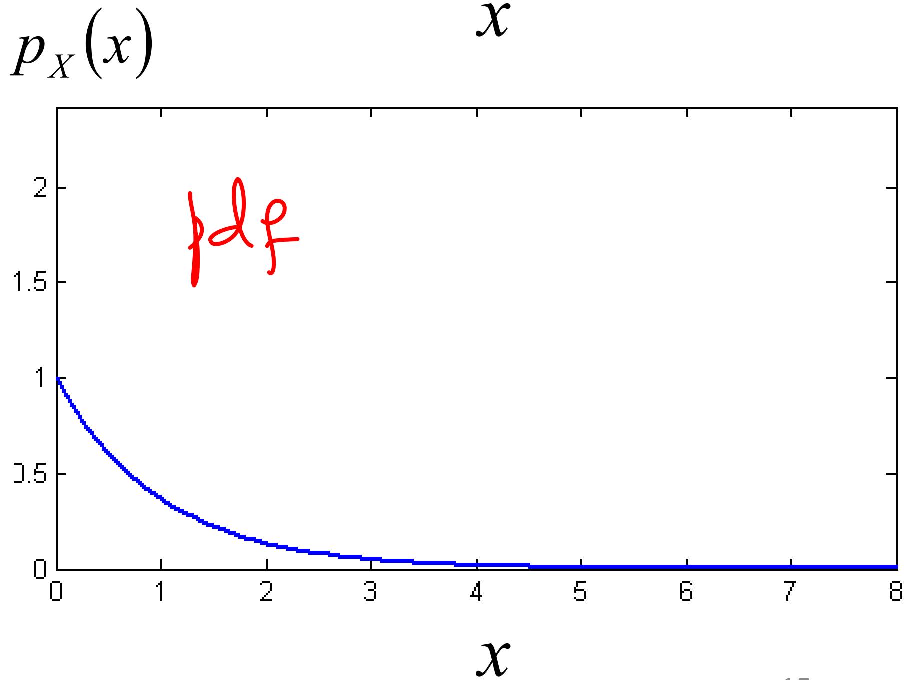
$$= \frac{P(t_0 < T^* \le t)}{P(T^* > t_0)} = \frac{\{1-\exp(-\lambda t_0)\} - \{1-\exp(-\lambda t)\}}{\exp(-\lambda t_0)};t \ge t_0$$

$\Rightarrow P_{T^*}(t \mid T^* > t_0) = 1 - \exp\{-\lambda(t - t_0)\};t_0 < t < \infty$

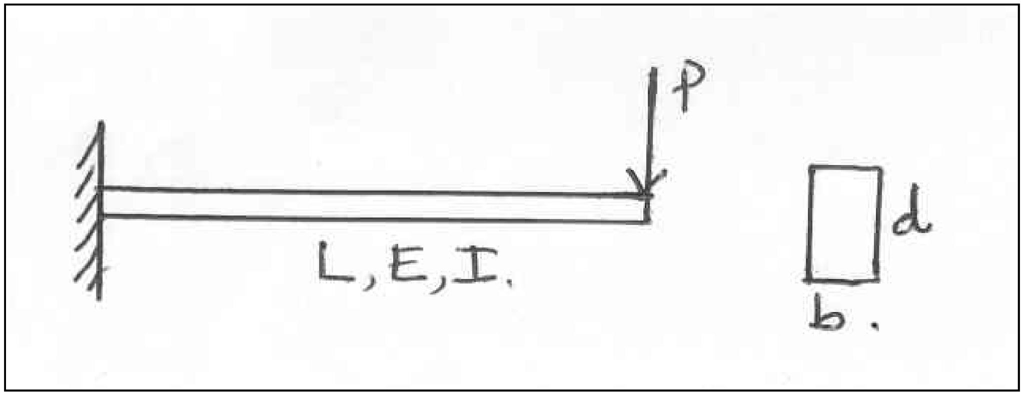
$\Rightarrow p_{T^*}(t \mid T^* > t_0) = \lambda \exp\{-\lambda(t - t_0)\};t_0 < t < \infty$

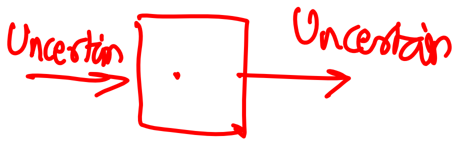
In words, failure to observe an event up to time t_0 does not alter one's prediction of the length of time (from t_0) before an event will occur. The future is not influenced by the past.





Transformation of random variables

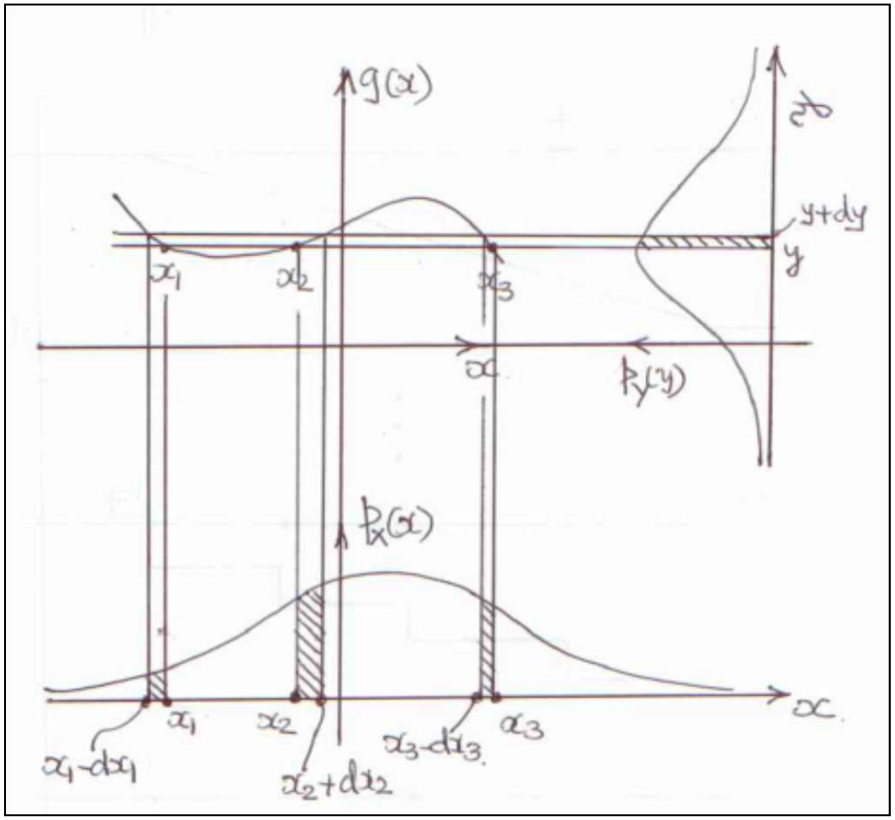




If *P,L,b,d,E* are RVs, what is the pdf of $\delta = \dfrac{PL^3}{3EI}$?

Basic problem of transformation of random variables

Let *X* be a RV; define *Y=g(X)*;
Given pdf of *X*, what is the pdf of *Y*?



$$P\left(y < Y \le y+dy\right) = p_Y\left(y\right)dy$$
$$= p_X\left(x_1\right)|\,dx_1\,| + p_X\left(x_2\right)|\,dx_2\,| + p_X\left(x_3\right)|\,dx_3\,|$$
$$\Rightarrow p_Y\left(y\right) = \frac{p_X\left(x_1\right)}{\left|\frac{dy}{dx}\right|_{x=x_1}} + \frac{p_X\left(x_2\right)}{\left|\frac{dy}{dx}\right|_{x=x_2}} + \frac{p_X\left(x_3\right)}{\left|\frac{dy}{dx}\right|_{x=x_3}}$$

In general

$$p_Y\left(y\right) = \sum_{i=1}^{n} \frac{p_X\left(x_i\right)}{\left|\frac{dy}{dx}\right|_{x=x_i}} \text{ with } x_i = g^{-1}\left(y\right)$$

Why modulus?

We wish to add probabilities.

Example 1

$$Y = \exp(X); \quad X \sim N(m, \sigma)$$

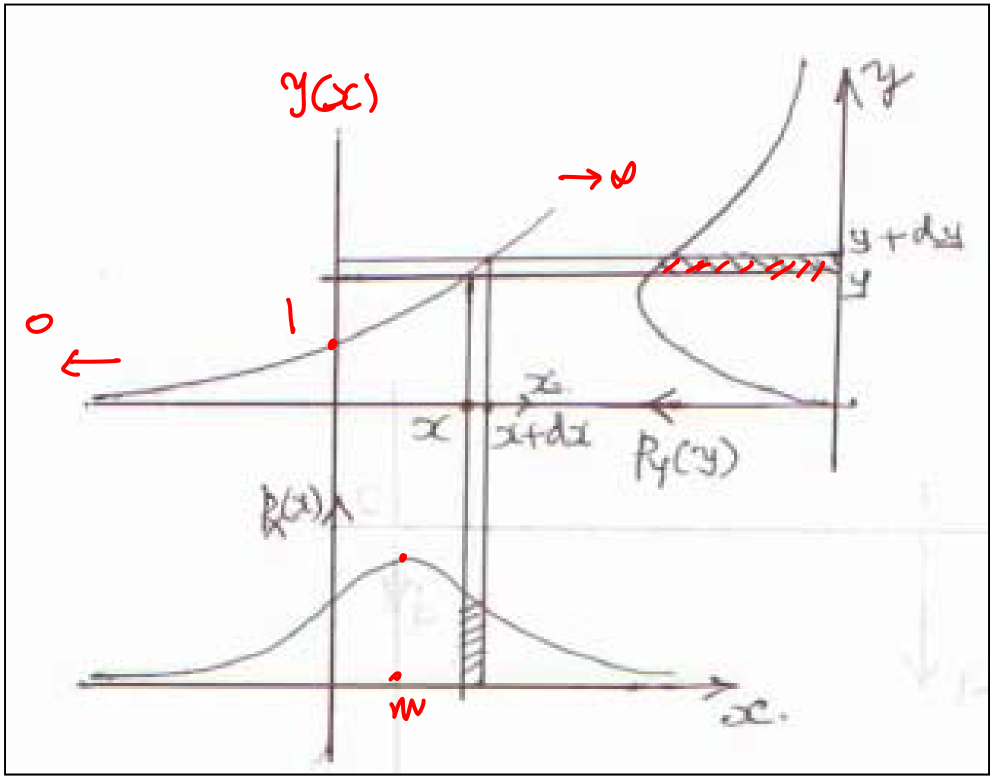
$$-\infty < x < \infty \Rightarrow 0 < y < \infty$$

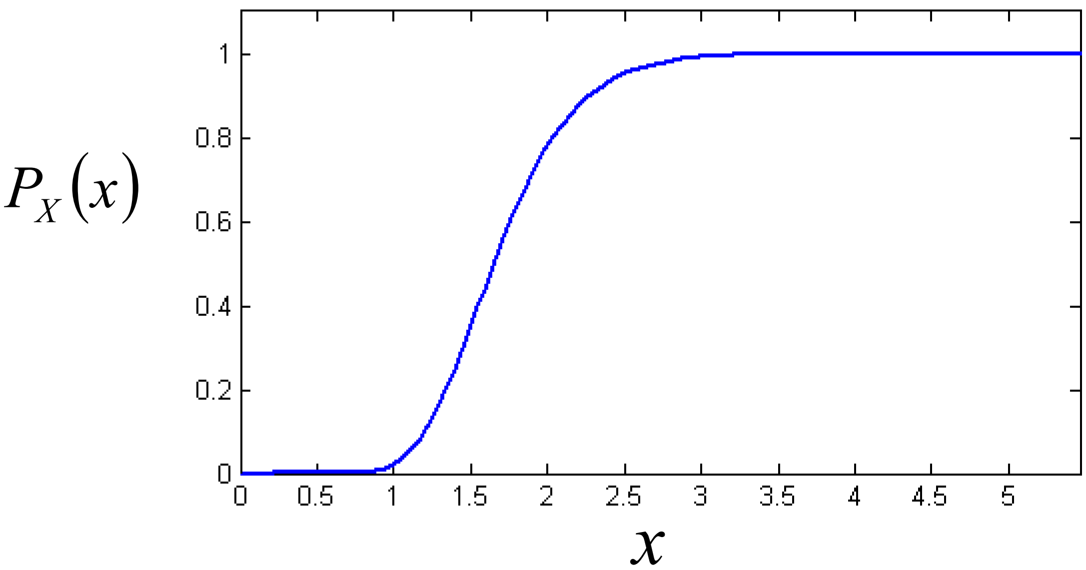
$$y = \exp(x) \Rightarrow x = \log y \;\&\; \frac{dy}{dx} = \exp(x) = y.$$

$$p_Y(y) = \left.\frac{p_X(x)}{\exp(x)}\right|_{x=\log y}$$

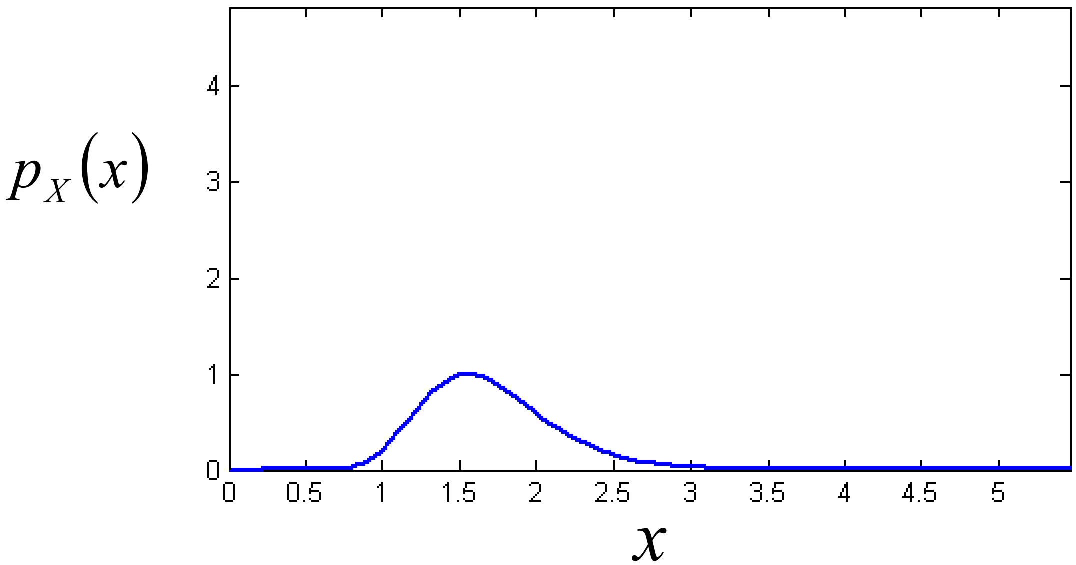
$$= \frac{1}{\sqrt{2\pi}\sigma y}\exp\left[-\frac{1}{2}\left(\frac{\log y - m}{\sigma}\right)^2\right]; 0 < y < \infty$$

Y is said to be lognormal distributed.





Lognormal Random variable

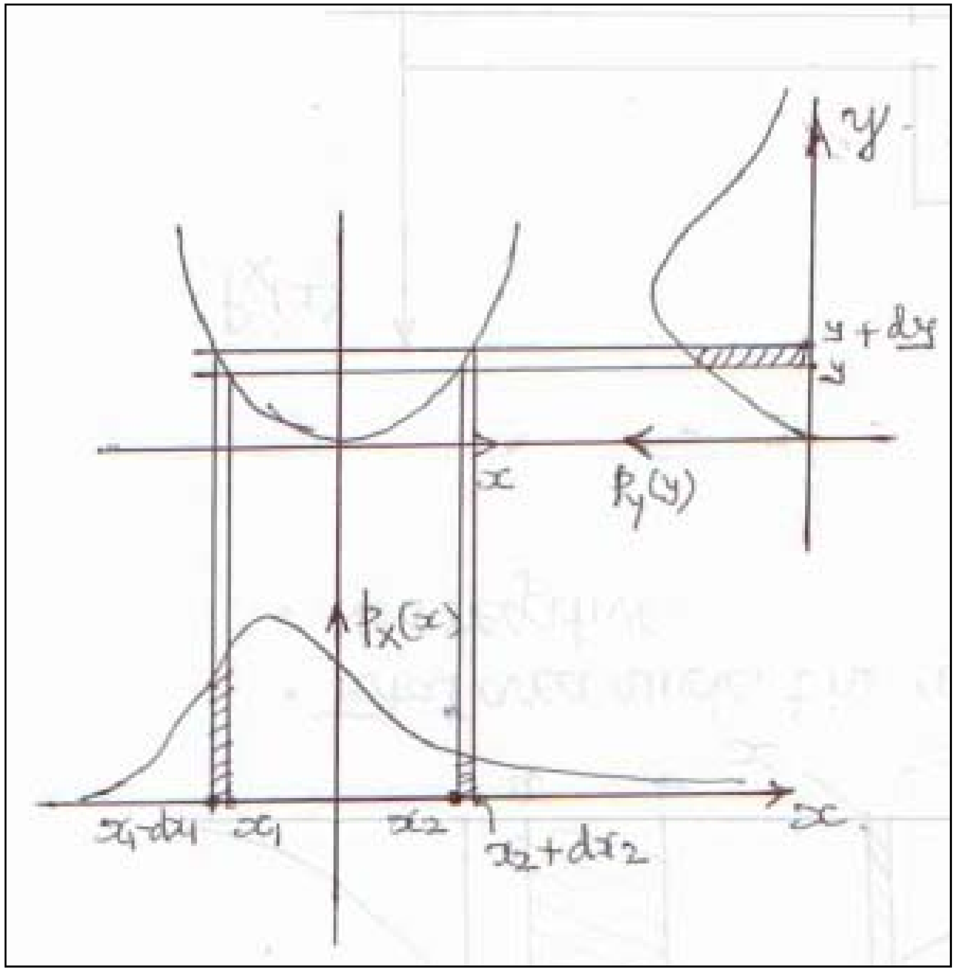


Example

$$Y = aX^2$$

$$\Rightarrow x = \pm\sqrt{\frac{y}{a}}; \quad \frac{dy}{dx} = 2ax$$

$$p_Y(y) = \frac{p_X\left(\sqrt{\frac{y}{a}}\right) + p_X\left(-\sqrt{\frac{y}{a}}\right)}{2\sqrt{ay}}; 0 < y < \infty$$



Example

$$U = \frac{X - m}{\sigma}; X \sim N(m, \sigma)$$

$$p_U(u) = \frac{p_X(\sigma u + m)}{\frac{1}{\sigma}} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right); -\infty < u < \infty$$

$\Rightarrow U \sim N(0,1)$

Remarks

(a) $U \sim N(0,1)$

(b) U is non-dimensional (why?)

(c) This is an illustration of a general result that Gaussian quantities are closed under linear operations.

Exercise

Let $X \sim N(m, \sigma)$. Define $Y = aX + b$.

Show that $Y \sim N(am + b, a\sigma)$.

Moments of a random variable: expectation operator

Let X be a random variable.

The complete description of X is through its PDF or pdf.

Questions

(a) Is pdf always required?

(b) Is pdf always available?

(c) Are simpler descriptions possible?

Mean of a random variable

$$\eta = \int_{-\infty}^{\infty} x p_X(x) dx \quad \text{(continuous random variable)}$$

$$\eta = \sum_{i=1}^{n} x_i P(X = x_i) \text{(discrete random variable)}$$

η is a measure of **central value** of X.

Is this notion consistent with our intutive notion of a mean?

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{i=1}^{r} n_i x_i$$

$$= \sum_{i=1}^{r} \left(\frac{n_i}{n} \right) x_i$$

$$= \sum_{i=1}^{r} P(X = x_i) x_i \sim \int_{-\infty}^{\infty} x p_X(x) dx$$

$$= \eta$$

Measure of dispersion : Variance and standard deviation

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\eta)^2 p_X(x)dx \text{ (continuous RV)}$$

$$\sigma^2 = \sum_{i=1}^{n} (x_i-\eta)^2 P(X = x_i) \text{ (discrete RV)}$$

$\sigma = +\sqrt{\sigma^2}$ is known as the standard deviation of X

Remarks

(a) σ^2 has the units of X^2.

(b) σ has the units of X (preferred).

(c) One could think of using

$\tilde{\sigma} = \int_{-\infty}^{\infty} |(x-\eta)| p_X(x) dx$ as a measure of dispersion.

The absolute operation is not amenable for mathematical manipulations and hence this definition is not used.

Mathematical expectation operator

Denoted by $\langle \bullet \rangle$ or $E[\bullet]$.

Definition

$$\langle g(X) \rangle = E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx$$

Remarks

(1) We will use $\langle \bullet \rangle$

(2) Let a be a constant. $\langle a \rangle = \int_{-\infty}^{\infty} a p_X(x) dx = a \int_{-\infty}^{\infty} p_X(x) dx = a.$

(3) Clearly, for $g(X) = X$, $\langle g(X) \rangle = \int_{-\infty}^{\infty} x p_X(x) dx = \eta$

(4) Similarly, for $g(X) = (X - \eta)^2$, $\langle g(X) \rangle = \int_{-\infty}^{\infty} (x - \eta)^2 p_X(x) dx = \sigma^2$

Remarks (continued)

(5) Define $m_n = \left\langle X^n \right\rangle = \int_{-\infty}^{\infty} x^n p_X(x)dx$; n-th order raw moments.

(6) Define $\mu_n = \left\langle (X-\eta)^n \right\rangle = \int_{-\infty}^{\infty} (x-\eta)^n p_X(x)dx$; n-th order central moment.

(7) COV = coefficient of variation = $\dfrac{\sigma}{\eta}$; $\eta \neq 0$.

(8) $\mu_0 = 1; \mu_1 = 0; \mu_2 = \sigma^2$

(9) skewness $= \dfrac{\mu_3}{\sigma^3}$; kurtosis $= \dfrac{\mu_4}{\sigma^4}$; (non-dimensional entities)

(10) $m_2 = \left\langle X^2 \right\rangle$ = mean square value; $\sqrt{m_2}$ = root mean square value.

(11) Root mean square value = standard deviation if mean = 0.

(12) $\left\langle (X-\eta)^2 \right\rangle = \left\langle X^2 + \eta^2 - 2X\eta \right\rangle = \left\langle X^2 \right\rangle + \eta^2 - 2\eta\left\langle X \right\rangle = \left\langle X^2 \right\rangle - \eta^2$

(13) $\sigma^2 + \eta^2 = \left\langle X^2 \right\rangle$. Clearly, $\sigma^2 > 0 \Rightarrow \eta^2 < \left\langle X^2 \right\rangle$. That is, $m_2 > m_1^2$.

Moments are not arbitrary. They need to satisfy certain inequalities.

(14) Two random variables may have the same mean and variance. This does not imply that they will have the same pdf.

Moment generating function

Definition : $\psi_X(s) = \langle \exp(sX) \rangle = \int_{-\infty}^{\infty} \exp(sx) p_X(x) dx$

That is $\psi_X(s)$ is the Laplace transform of the pdf.

$$\exp(sX) = 1 + sX + \frac{(sX)^2}{2!} + \frac{(sX)^3}{3!} + \cdots$$

$$\psi_X(s) = 1 + s\langle X \rangle + \frac{s^2}{2!}\langle X^2 \rangle + \frac{s^3}{3!}\langle X^3 \rangle + \cdots$$

$$\Rightarrow \left.\frac{d\psi_X}{ds}\right|_{s=0} = \langle X \rangle; \left.\frac{d^2\psi_X}{ds^2}\right|_{s=0} = \langle X^2 \rangle; \cdots \left.\frac{d^n\psi_X}{ds^n}\right|_{s=0} = \langle X^n \rangle$$

Characteristic function

Definition: $\varphi_X(\omega) = \langle \exp(i\omega X) \rangle = \int_{-\infty}^{\infty} \exp(i\omega x) p_X(x) dx$

Here ω is real valued. Thus, $\varphi_X(\omega)$ is the Fourier transform of the pdf.

It can be shown that $\frac{1}{i^n} \frac{d^n \varphi_X}{d\omega^n}\bigg|_{\omega=0} = \langle X^n \rangle$.

By using inverse Fourier transform, it follows that

$p_X(x) = \int_{-\infty}^{\infty} \varphi_X(\omega) \exp(-i\omega x) dx.$

Complete specification of a RV

- Specification of the probability space.
- PDF
- pdf
- Moment generating function
- Characteristic function
- Moments of all orders

Bernoulli random variable

$$P(X=0)=p;\ P(X=1)=1-p$$

$$\langle X\rangle=\sum_{i=1}^{2}x_iP(X=x_i)=0\times p+1\times(1-p)=1-p \quad m_1$$

$$\langle X^2\rangle=\sum_{i=1}^{2}x_i^2P(X=x_i)=0\times p+1\times(1-p)=1-p \quad m_2$$

$$\sigma_X^2=\langle X^2\rangle-\eta^2=(1-p)-(1-p)^2=p(1-p)$$

$$\phi_X(\omega)=\exp(i\omega 0)p+\exp(i\omega)(1-p)=p+(1-p)\exp(i\omega)$$

$$\langle X^n\rangle=1-p$$

Poisson random variable

$$P\left(X=k\right)=\exp(-a)\frac{a^k}{k!};k=0,1,2,\cdots,$$

$$\sum_{k=0}^{\infty}\exp(-a)\frac{a^k}{k!}=\exp(-a)\exp(a)=1$$

$$\left\langle X\right\rangle=\sum_{k=0}^{\infty}k\exp(-a)\frac{a^k}{k!}=\sum_{k=1}^{\infty}k\exp(-a)\frac{a^k}{k!}$$

$$=\sum_{k=1}^{\infty}k\exp(-a)a\frac{a^{k-1}}{k(k-1)!}$$

$$=a\sum_{k=1}^{\infty}\exp(-a)\frac{a^{k-1}}{(k-1)!}\quad(\text{put } n=k\text{-}1)$$

$$=a\sum_{n=0}^{\infty}\exp(-a)\frac{a^n}{n!}=a\exp(-a)\exp(a)=a$$

Variance

$$\left\langle X^2 \right\rangle = \sum_{k=0}^{\infty} k^2 \exp(-a) \frac{a^k}{k!}$$

$$= \sum_{k=1}^{\infty} k^2 \exp(-a) \frac{a^k}{k!} = \sum_{k=0}^{\infty} k^2 \exp(-a) a^2 \frac{a^{k-2}}{k(k-1)(k-2)!}$$

$$= a^2 + a$$

$$\sigma_X^2 = a^2 + a - a^2 = a$$

Characteristic function

$$\varphi_X(\omega) = \left\langle \exp\left(i\omega X\right)\right\rangle$$

$$= \sum_{k=0}^{\infty} \exp(i\omega k)\exp(-a)\frac{a^k}{k!}$$

$$= \sum_{k=0}^{\infty} \exp(-a)\frac{\left(a\exp\left(i\omega\right)\right)^k}{k!}$$

$$= \exp\left[a\left\{\exp\left(i\omega\right)-1\right\}\right]$$

Gaussian random variable

$$N(m,\sigma) \Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]; -\infty < x < \infty$$

Area under the curve (=1?)

$$\int_{-\infty}^{\infty} p_X(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] dx$$

Substitute $u = \left(\frac{x-m}{\sigma}\right) \Rightarrow \sigma du = dx$

$$\Rightarrow \int_{-\infty}^{\infty} p_X(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}u^2\right] du$$

Let I= $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}u^2\right] du$

$$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{u^2+v^2}{2}\right)\right] du\,dv$$

Substitute $u = r\cos\theta; v = r\sin\theta \Rightarrow r\,dr\,d\theta = du\,dv$

$$\Rightarrow I^2 = \int_0^{\infty}\int_0^{2\pi} r \exp\left[-\frac{r^2}{2}\right] dr\,d\theta = 2\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} p_X(x)\,dx = 1$$

Exercise

Show that

$$\langle X \rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = m$$

$$\left\langle (X - m)^2 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-m)^2 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = \sigma^2$$

$$\left\langle (X - m)^3 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-m)^3 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = 0$$

$$\left\langle (X - m)^4 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-m)^4 \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \right] dx = 3\sigma^4$$

Skewness=0

Kurtosis=3

$$\phi_X(\omega) = \exp\left(im\omega - \frac{1}{2}\sigma^2\omega^2 \right)$$

Remark

A Gaussian random variable is completely specified in terms of its mean and standard deviation.

More on Gaussian random variable

Let $X \sim N(m, \sigma)$.

$$\Rightarrow U = \frac{X - m}{\sigma} \sim N(0,1).$$

$$\Rightarrow p_U(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{u^2}{2}\right] du$$

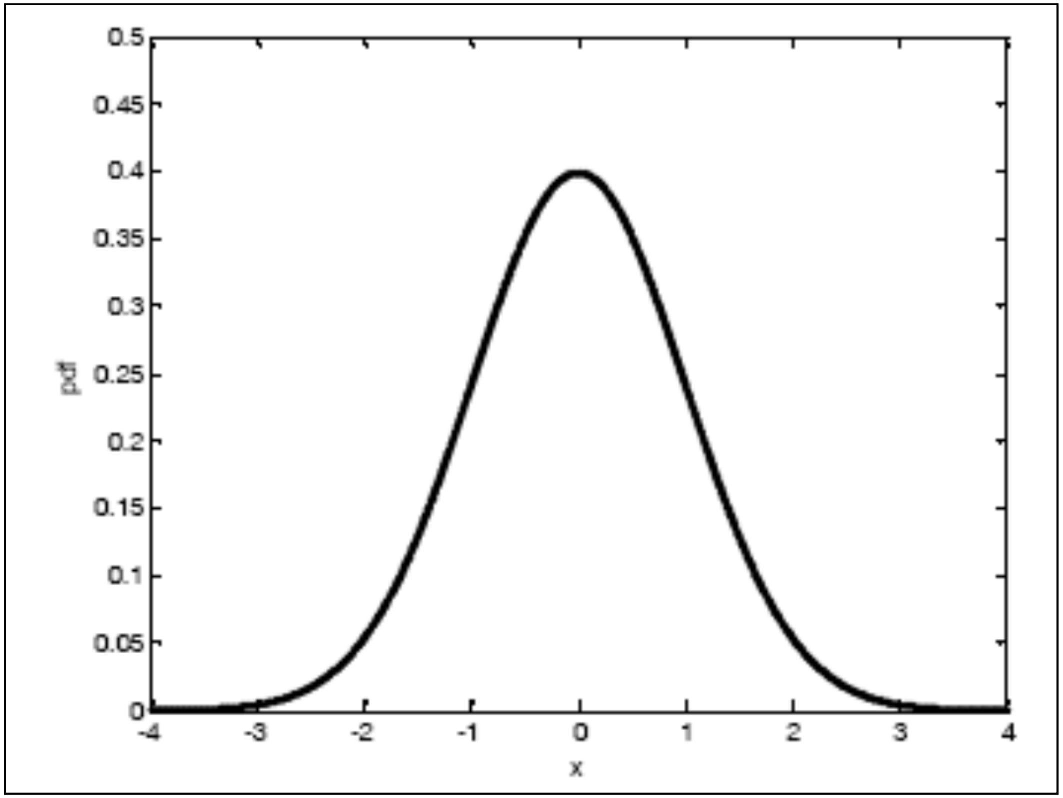
$$\Rightarrow \langle U \rangle = 0 \ \& \ \langle U^2 \rangle = 1.$$

$$P_U(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left[-\frac{s^2}{2}\right] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left[-\frac{s^2}{2}\right] ds + \frac{1}{\sqrt{2\pi}} \int_{0}^{u} \exp\left[-\frac{s^2}{2}\right] ds$$

$$= 0.5 + \mathrm{erf}(u)$$

$$\mathrm{erf(u)} = \frac{1}{\sqrt{2\pi}} \int_{0}^{u} \exp\left[-\frac{s^2}{2}\right] ds$$



$$\int_{-1}^{1} p_U(u)du = 0.68268$$

$$\int_{-2}^{2} p_U(u)du = 0.95450$$

$$\int_{-3}^{3} p_U(u)du = 0.99730$$

$$\Rightarrow P(m - 3\sigma < X \leq m + 3\sigma) = 0.99730$$

$m \pm 3\sigma$ ~ extremes

40

Time to the k - th event : Gamma distribution.

Define $X_k =$ time for the k-th arrival of a Poisson event.

We know that the times between arrivals, $T_i \; i = 1, 2, \cdots, k$ are independent and have exponential distribution.

$$\Rightarrow X_k = \sum_{i=1}^{k} T_i$$

$$\Rightarrow X_1 = T_1 \Rightarrow p_{X_1}(x) = \lambda \exp(-\lambda x); x \geq 0$$

$$X_2 = X_1 + T_2 \Rightarrow p_{X_2}(x) = \int_0^x p_{X_1}(x-u)\, p_{T_2}(u) du$$

($Note$: The upper limit here is not $\infty \because p_{X_1}(x-u) = 0$ for $x - u < 0$.

$$p_{X_2}(x) = \int_0^x \lambda \exp\{-\lambda(x-u)\} \lambda \exp(-\lambda u)\, du = \lambda^2 x \exp(-\lambda x); x \geq 0$$

$$X_3 = X_2 + T_3 \Rightarrow \int_0^x p_{X_1}(x-u)\, p_{T_2}(u) du$$

$$p_{X_3}(x) = \int_0^x \lambda^2 (x-u) \exp\{-\lambda(x-u)\} \lambda \exp(-\lambda u)\, du = \frac{\lambda(\lambda x)^2 \exp(-\lambda x)}{(3-1)!}$$

$$\Rightarrow p_{X_k}(x) = \frac{\lambda(\lambda x)^{k-1} \exp(-\lambda x)}{(k-1)!}; x \geq 0.$$

(Why the name "gamma"?)

Show that

$$\langle X_k \rangle = \frac{k}{\lambda} \;\&\; \sigma^2_{X_k} = \frac{k}{\lambda^2}$$

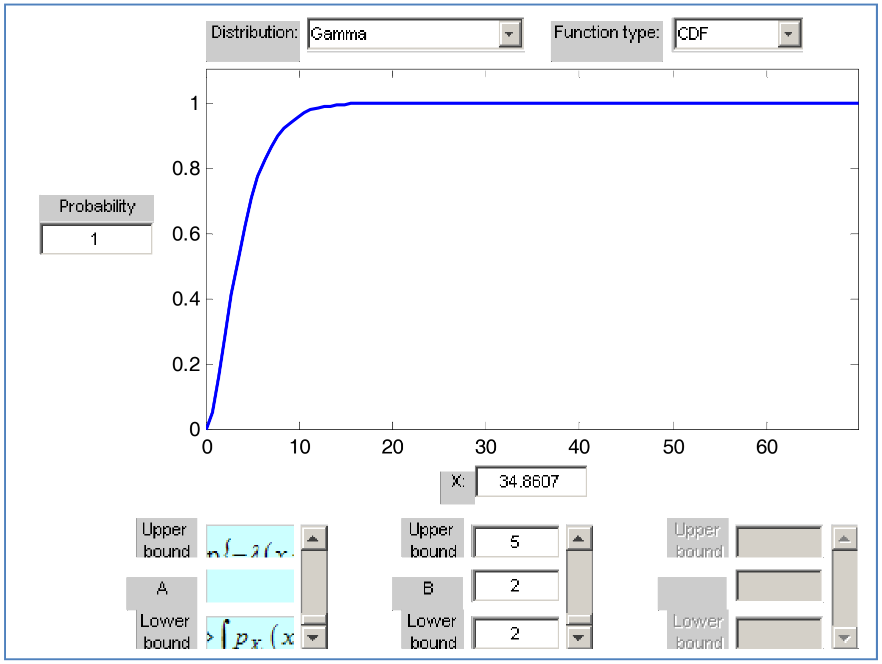
General form of gamma distribution

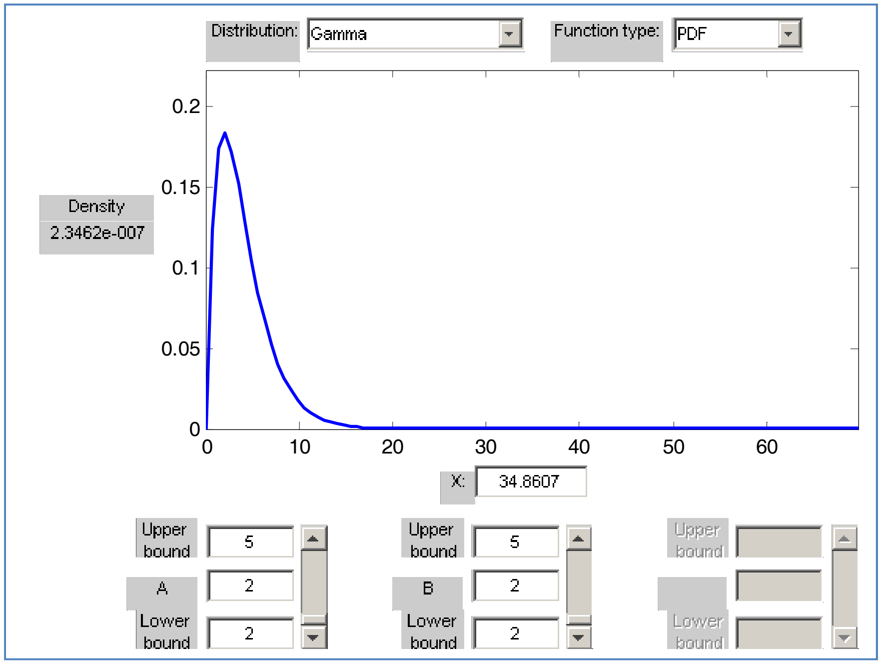
Replace factorial by gamma function

$$p_X(x) = \frac{\lambda(\lambda x)^{k-1} \exp(-\lambda x)}{\Gamma(k)}; x \geq 0$$

Here $\lambda > 0 \;\&\; k > 0$

(k is not necessarily an integer)


Distribution: Gamma
Function type: CDF
Probability
1
X: 34.8607
Upper bound
A
Lower bound
Upper bound
5
B
2
Lower bound
2
Upper bound
Lower bound


Distribution: Gamma
Function type: PDF
Density
2.3462e-007
X: 34.8607
Upper bound
5
A
2
Lower bound
2
Upper bound
5
B
2
Lower bound
2
Upper bound
Lower bound

Reccurrence interval and Return period

Consider the occurrence of earthquakes of $M > m^*$ at a given location.

Consider time meaured in number of years (in integers).

Let the PDF of M be $P_M\left(m\right)$.

$\Rightarrow P\left(M > m^*\right) = 1 - P_M\left(m^*\right)$.

Consider the occurrence of earthquakes of $M > m^*$ as a Bernoulli sequence.

It follows, $p = 1 - P_M\left(m^*\right)$.

Let N=number of years for the first occurrence of earthquake with $M > m^*$.

$N =$ recurrence time.

$P\left(N = k\right) = \left(1 - p\right)^{k-1} p; k = 1, 2, 3, \cdots\infty$ (Geometric distribution).

It can be proved that $\langle \mathrm{N} \rangle = \dfrac{1}{p}$.

Thus if $p = 0.01$, $\langle N \rangle = 100$ years; m^* is called the 100 years earthquake.

$P\left(N > 10\right) = 1 - P\left(N \leq 10\right) =$ P(no earthquake of $M > m^*$ occurs in 10 years)

$= 1 - \left\{1 - \left(1 - p\right)^{10}\right\} = 0.92$

Similarly, we get $P(N > 30) = 0.74$.

Example

The yearly maximum wind speed follows Type I extreme value distribution with parameters u=97.6 kmph and α=0.066. Determine the return period of the wind speed of 158.1 kmph. Also determine the 20 year return period wind speed.

Solution

Part - 1

$$P_V(v) = \exp\left[-\exp\left\{-\alpha(v-u)\right\}\right]; -\infty < v < \infty$$

$$p = 1 - P_V(158.1) = 1 - 0.9817$$

$$\langle \mathrm{N} \rangle = \frac{1}{1-0.9817} = 54.7 \text{ years.}$$

Part - 2

$$p = \frac{1}{20} = 0.05 \Rightarrow P_V(v) = 1 - 0.05 = 0.95$$

$$\Rightarrow \exp\left[-\exp\left\{-\alpha(v-u)\right\}\right] = 0.95$$

$$\Rightarrow v = 142.6 \text{ kmph}$$