Stochastic Structural Dynamics

Lecture-1
Definition of probability measure and conditional probability

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This Lecture

- What is this course about?
- Begin reviewing theory of probability

Loads on engineering structures

- Earthquake
- Wind
- Waves
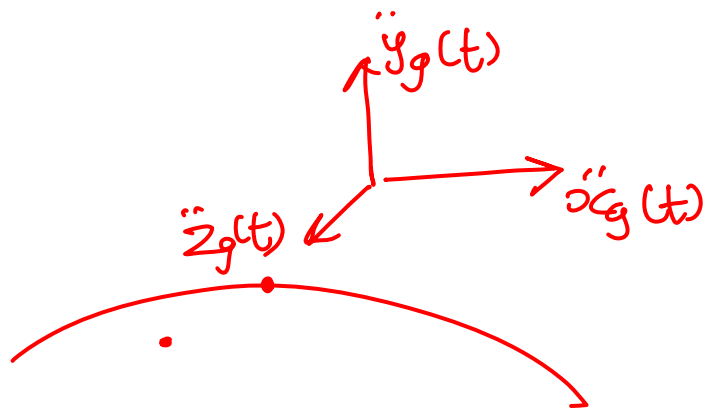- Guideway unevenness
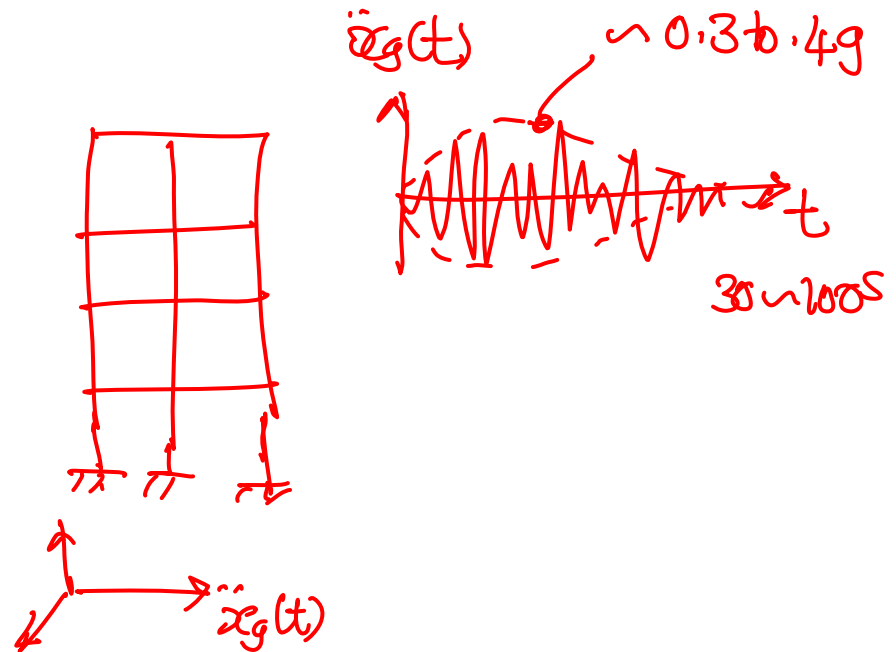- Traffic

•Dynamic
•Random

Loads on engineering structures

- Earthquake

Loads on engineering structures

- Wind

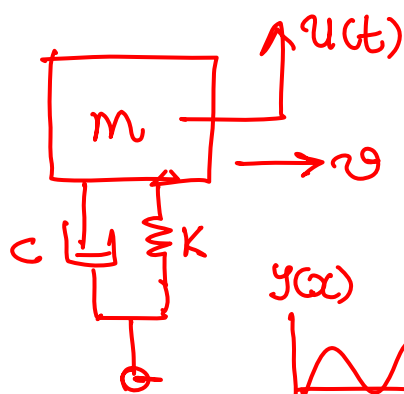# Loads on engineering structures

- Guideway unevenness

$$m\ddot{u} + c\frac{d}{dt}\left[u(t) - y(vt)\right]$$

$$+ K\left[u(t) - y(vt)\right] = 0$$
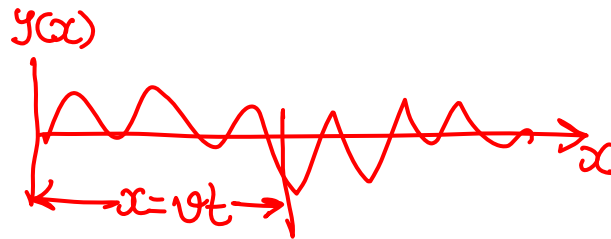
$$m\ddot{u} + c\dot{u} + Ku$$

$$= c\frac{d}{dt}\left[y(vt)\right]$$

$$+ K y(vt)$$

$$= f(t)$$

Uncertainties in structural engineering problems

- **Loads** (earthquakes, wind, waves, guide way unevenness…)
- **Structural properties** (elastic constants, mass, damping, strength, boundary conditions, joints…)
- **Modeling** (analytical, computational and experimental)
- **Condition assessment in existing structures**
- *Human errors*

Stochastic structural dynamics

•Branch of structural dynamics in which the uncertainties in loads are quantified mathematically using theory of probability, random processes and statistics.

    •**Random vibration analysis; probabilistic structural dynamics**

•Failure of structures under uncertain dynamic loads

•Design of structures under uncertain dynamic loads

Also important in experimental vibration analysis

    •Measurement of frequency response and impulse response functions

    •Seismic qualification testing

    •Condition assessment of existing structures

Pre-requisites

Basic background in

•Linear vibration analysis

•Probability and statistics

Mathematical models for uncertainty:

- **Probability, random variables, random processes, statistics**.
- Fuzzy logic.
- Interval algebra.
- Convex models.

Review of probability and random processes

Suggested books

1. A Papoulis and S U Pillai, 2006, Probability, random variables and stochastic processes, 4th Edition , McGraw Hill, Boston.
2. J R Benjamin and C A Cornell, 1970, Probability, statistics, and decision for civil engineers, McGraw Hill Book Company, Boston.

Definitions of probability:

1. Classical definition
2. Relative frequency
3. Axiomatic

Classical (mathematical or _a priori_) definition:

If a random experiment can result in _n_ outcomes, such that these outcomes are
- **equally likely**
- **mutually exclusive**
- **collectively exhaustive**

and, if out of these n outcomes, _m_ are favourable to the occurrence of an event A, then the probability of the event A is given by P(A)=_m/n_.

Example: P(getting even number on die tossing)=3/6=1/2.

Objections
- **What is "equally likely"?**
- **What if not equally likely? (what is the probability that sun would rise tomorrow?)**
- **No room for experimentation.**
- **Probability is required to be a rational number.**

Relative frequency (posteriori) definition

If a random experiment has been performed *n* number of times and if *m* outcomes are favorable to event A, then the probability of event A is given by

$$P(A) = \lim_{n \to \infty} \frac{m}{n}.$$

Objections

- What is meant by limit here?
- One cannot talk about probability without conducting an experiment.
  - **What is the probability that someone meets with an accident tomorrow?**
- Probability is required to be a rational number.

Example:

Toss a die 1000 times; note down how many times an even number turns up (say, 548).

P(even number)=548/1000.

N=1000 here is deemed to be sufficiently large.

There is no guarantee that as the
number of trials increases, the probability would converge.
The die need not be "fair".

Axiomatic definition

- **Undefined notions (primitives)**
  - Experiments
  - Trials
  - Outcomes
- **An experiment is a physical phenomenon that can be observed repeatedly. A single performance of an experiment is a trial. The observation made on a trial is its outcome.**
- **Axioms are statements that are commensurate with our experience. No proofs exist. All truths are relative to the accepted axioms.**

- **Random experiment** $(\mathcal{E})$

  is an experiment such that
    - the outcome of a specific trial cannot be predicted, and
    - it is possible to predict all possible outcomes of any trial.

Remarks

- $\mathcal{E}$: the first technical term.
- Example: Toss a coin. We know that we will either get head or tail. In any given trial however we do not know before hand what would be the outcome.
- What cannot be envisaged, does not enter the theory.

Axiomatic definition (continued)

Sample space (Ω)

Set of all possible outcomes of a random experiment.

Examples

(1) Coin tossing: $\Omega = \begin{pmatrix} h & t \end{pmatrix}$; Cardinality=2; finite sample space.

(2) Die tossing: $\Omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$; Cardinality=6; finite sample space.

(3) Die tossing till head appears for the first time:

$\Omega = \begin{pmatrix} h & th & tth & ttth & tttth & \cdots \end{pmatrix}$; Cardinality=$\infty$; countably infinite sample space.

(4) Maximum rainfall in a year: $\Omega = \begin{pmatrix} 0 \le X < \infty \end{pmatrix}$;

Cardinality=∞; uncountably infinite sample space.

Elements of Ω are called sample points.

Ω can be thought of as outcome space.

Consider a set with n elements.

Number of subsets $= {}^nC_o + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = (1+1)^n = 2^n$

Axiomatic definition (continued)

Event space (B)

Ω is finite : B is the set of all subsets of Ω.

Ex : $\Omega = \begin{pmatrix} h & t \end{pmatrix}; B = \begin{pmatrix} h & t & \Omega & \phi \end{pmatrix}$

Cardinality of $B = 2^N$; N = cardinality of Ω.

In general : B is the sigma algebra of subsets of Ω

Definition

Let C be a class of subsets of Ω. If

(a) $A \in \Omega \Rightarrow A^C \in \Omega, \&$

(b) $\{A_i\}_{i=1}^{\infty} \in \Omega \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Omega$

then, we say that C is a sigma algebra of subsets of Ω.

Elements of B are known as events

Axiomatic definition (continued)

Probability (P)

$P : B \rightarrow [0,1]$ such that

$\text{Axiom 1 (axiom of non-negativity)} \; P(A) \geq 0 \; \forall A \in B$

$\text{Axiom 2 (axiom of normalization)} \; P(\Omega) = 1$

Axiom 3 (axiom of additivity)

$$\{A_i\}_{i=1}^{\infty} \in \Omega, \; A_i \cap A_j = \phi \; \forall i \neq j \Rightarrow P\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P(A_i)$$

Probability space is the ordered triplet : $\langle \Omega, B, P \rangle$

Note:

We wish to assign probability to not only to elementary events
(elements of sample space) but also to compound events
(subsets of sample space).

When sample space is not finite, (as when it is the real line)
there exists subsets of sample space which cannot be
expressed as countable union and intersections of intervals.

On such events we will not be able to assign probabilities consistent
with the third axiom.

To overcome this difficulty we exclude these events from the event space.

Ω
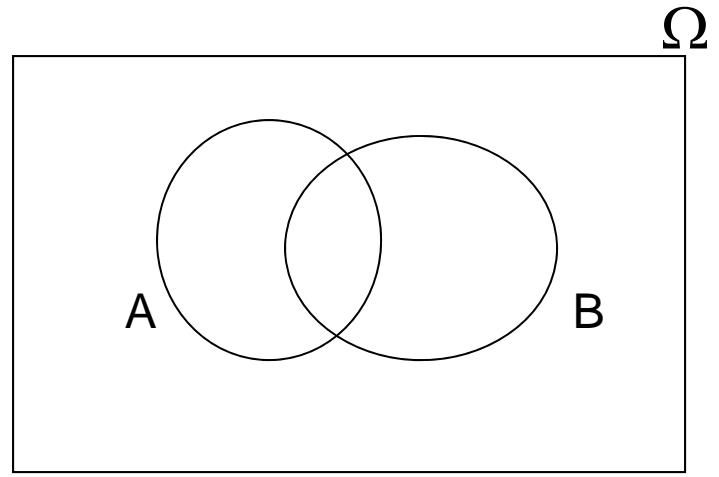
Remarks

(1) $P(A^c) = 1 - P(A)$

(2) $A \subset B \Rightarrow P(A) \le P(B)$

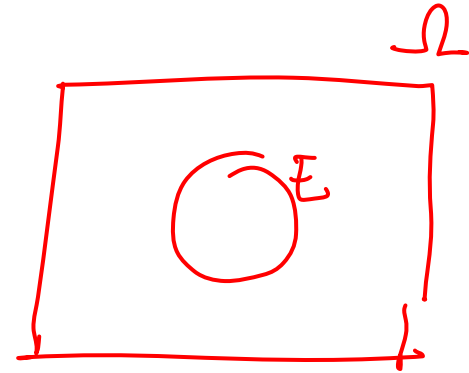(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(4) $P(\phi) = 0$

Proof 1

$E \cup E^c = \Omega; \quad E \cap E^c = \phi$

$P(E \cup E^c) = P(E) + P(E^c) \, (\text{Axiom } 3)$

$\Rightarrow P(\Omega) = P(E) + P(E^c) = 1 \, (\text{Axiom } 2)$

$\Rightarrow P(E^c) = 1 - P(E)$

Proof 4 : Use proof 1 with $E = \Omega$;

Note : $\Omega^c = \phi$.

Proof 3

$A \cup B = A \cup (A^c \cap B); \quad A \cap (A^c \cap B) = \phi$

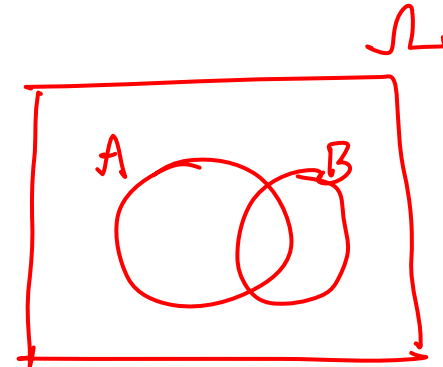$\Rightarrow P(A \cup B) = P(A) + P(A^c \cap B) \text{(Axiom 3)} \cdots (1)$

$B = (A \cap B) \cup (A^c \cap B); \quad (A \cap B) \cap (A^c \cap B) = \phi$

$\Rightarrow P(B) = P(A \cap B) + P(A^c \cap B) \text{(Axiom 3)} \cdots (2)$

From (1) and (2)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\text{Proof } 2 : \text{hint} : \text{Use proof } 3.$

Conditional probability and stochastic independence

Definition

$P(A|B) =$ Probability of event A given that B has occurred

$$= \frac{P(A \cap B)}{P(B)}; P(B) \neq 0.$$

Example: Fair Die tossing

$\Omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$

The die has been tossed and an even number has been observed.

$P(2|\text{Even}) = ?$

Approach 1: Even $= \begin{pmatrix} 2 & 4 & 6 \end{pmatrix}$

$P(2|\text{Even}) = 1/3$ (Classical definition)

Approach 2:

$P(2|\text{Even}) = \dfrac{P(2 \cap \text{Even})}{P(\text{Even})}$

$(2 \cap \text{Even}) = (2)$

$\Rightarrow P(2|\text{Even}) = \dfrac{1/6}{1/2} = 1/3.$

Conditional Probability obeys all the axioms

(1) $P(A|B) \geq 0$

(2) $P(\Omega|B) = 1$

(3) $P(A \cup C|B) = P(A|B) + P(C|B)$ if $A \cap C = \phi$

Stochastic independence

Events A and B are said to be stochastically independent
if any one of the following four statements is true:

(1) The probability of occurrence of event A is not affected
by the occurrence of event B.

(2) $P(A \cap B) = P(A)P(B)$

(3) $P(A \mid B) = P(A)$

(4) $\dfrac{P(A \cap B)}{P(B)} = P(A); P(B) \neq 0.$

Notation : A and B are independent

$A \perp B \Rightarrow P(A \cap B) = P(A)P(B)$

Remarks

(1) Defintion 1 is not useful to verify if A and B are independent.

(2) If we need to verify if A and B are independent, we need to
find $P(A), P(B), P(B \mid A), \& P(A \cap B)$ and use defintions 2,3, or 4.

(3) Independence of more than two events can also be defined. Thus
$\{A_i\}_{i=1}^{3}$ are said to be independent if

(1) $P(A_i \cap A_j) = P(A_i)P(A_j) \forall i, j = 1, 2, 3 \& i \neq j$, and

(2) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

Example

Toss two coins.

$\Omega = \begin{pmatrix} hh & ht & th & tt \end{pmatrix}$

Let $a, b \geq 0$, such that $(a+b) = 1$.

Let $P(hh) = a^2 \quad P(tt) = b^2 \quad P(ht) = P(th) = ab.$

Clearly $P(\Omega) = P(hh) + P(tt) + P(ht) + P(th) = (a+b)^2 = 1.$

Define two events

$E_1 = \text{head on the first coint} = \begin{pmatrix} hh & ht \end{pmatrix}$

$E_2 = \text{head on the second coint} = \begin{pmatrix} hh & th \end{pmatrix}$

$Question:$ verify if $E_1 \& E_2$ are independent.

$P(E_1) = P\begin{pmatrix} hh & ht \end{pmatrix} = a^2 + ab = a(a+b) = a$

$P(E_2) = P\begin{pmatrix} hh & th \end{pmatrix} = a^2 + ab = a(a+b) = a$

$P(E_1 \cap E_2) = P(hh) = a^2$

$\Rightarrow P(E_1 \cap E_2) = P(E_1)P(E_2)$

$\Rightarrow E_1 \& E_2$ are independent.

Example: in which three events are pairwise independent but are not independent.

Consider a fair tetrahedron (this has four faces)

Let the four faces be painted as

Green, Yellow, Black and G+Y+B.

$$P(Y) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}; P(G) = P(B) = \frac{1}{2};$$

$$P(GY) = \frac{1}{4} = P(G)P(Y)$$

$$P(GB) = \frac{1}{4} = P(G)P(B)$$

$$P(YB) = \frac{1}{4} = P(Y)P(B)$$

$$P(GYB) = \frac{1}{4} \neq P(G)P(Y)P(B) = \frac{1}{8}.$$

Example

Consider a random experiment involving tossing of two dies. Define

A = (even on die 1)

B = (even on die 2)

C = (sum of numbers on die 1 and die 2)

Examine if A, B, and C are independent.

Total probability theorem

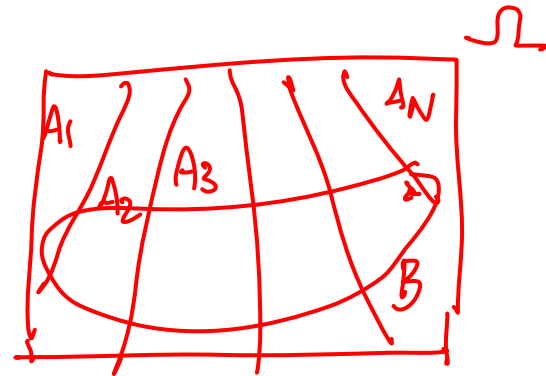Let $\{A_i\}_{i=1}^{N}$ constitute a partition of Ω.

That is, $\bigcup_{i=1}^{N} A_i = \Omega; \quad A_i \cap A_j = \phi \, \forall \, i \neq j.$

Let B be a set.

$$B = \bigcup_{i=1}^{N} (A_i \cap B)$$

$$P(B) = P\left(\bigcup_{i=1}^{N} (A_i \cap B) \right) = \sum_{i=1}^{N} P(A_i \cap B)$$

$$P(B) = \sum_{i=1}^{N} P(B \mid A_i) P(A_i) \, \mathbin{/\!/}$$

$$P(A_i \cap B) = P(B \mid A_i) \, P(A_i)$$

Bayes' theorem

$$P\left(A_i \mid B\right) = \frac{P\left(A_i \cap B\right)}{P(B)} = \frac{P\left(B \mid A_i\right)P(A_i)}{P(B)}$$

$$P\left(A_i \mid B\right) = \frac{P\left(B \mid A_i\right)P(A_i)}{\sum_{i=1}^{N} P\left(B \mid A_i\right)P(A_i)}$$

$P(A_i) = \text{a priori probability}$

$P(A_i \mid B) = \text{posteriori probability}$