INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

Lecture -35

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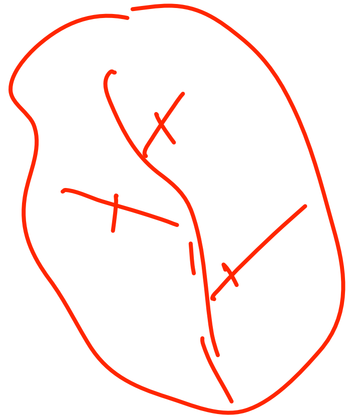
Summary of the previous lecture



- Multivariate stochastic models
  - Multisite Markov model

$$X_{t+1} = EX_t + G\varepsilon$$

where
X_t is a p x 1 vector of standardized values of the variable generated at time t,
E is a p x p diagonal matrix whose j^{th} diagonal element is $\rho_j(1)$,
G is a p x p diagonal matrix whose j^{th} diagonal element is $\sqrt{1-\rho_j^2(1)}$
ε is a p x 1 vector of random variates

  - Matalas model

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

Multivariate Stochastic models

- Matalas (1967) has given a multisite normal generation model that preserves the mean, variance, lag one serial correlation, lag one cross-correlation and lag zero cross-correlation.

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

Standardized data

$$\hat{x}_t = \frac{x_t - \bar{x}}{s}$$

where

X_t and X_{t+1} are p x 1 vectors representing standardized data corresponding to p sites at time steps t and $t+1$ resp.

Assumption is that the model is multivariate normal.

Ref.: Matalas, N.C. (1967) Mathematical assessment of synthetic hydrology, Water Resources Research 3(4):937-945

Multivariate Stochastic models

ε_{t+1} is N(0,1); p x 1 vector with ε_{t+1} independent of X_t.

A and B are coefficient matrices of size p x p. B is assumed to be lower triangular matrix

$$M_0 = E\left[X_t X_t^{'}\right]$$

M_0 is the cross-correlation matrix (size pxp) of lag zero

$$m_0\left(i,j\right) = \frac{1}{n}\sum_{t=1}^{n}\left(\frac{Q_{i,t} - \bar{Q}_i}{s_i}\right)\left(\frac{Q_{j,t} - \bar{Q}_j}{s_j}\right)$$

Multivariate Stochastic models

- The expectation of $X_t X_{t-1}'$ is denoted by M_1

$$M_1 = E\left[X_t X_{t-1}'\right]$$

If $m_1(i\ ,j)$ is a element of M_1 matrix (size p x p) in the i^{th} row and j^{th} column,

$$m_1(i,j) = E\left[X(i,t)X(j,t-1)\right]$$

Expected value of a matrix is matrix of expected values of individual elements

$$m_1(i,j) = \frac{1}{n-1}\sum_{t=2}^{n} x(i,t)x(j,t-1)$$

n is no. of time periods

Multivariate Stochastic models

$$m_1(i,j) = \frac{1}{n-1}\sum_{t=2}^{n}\left(\frac{Q_{i,t} - \bar{Q}_i}{s_i}\right)\left(\frac{Q_{j,t-1} - \bar{Q}_j}{s_j}\right)$$

Q is the original random variable before standardization e.g., stream flow

i.e., $m_1(i\ ,j)$ represents lag one cross correlation between the data at sites i and j .

Therefore M_1 is the cross-correlation matrix of lag one.

Multivariate Stochastic models

Considering the model,

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

Post multiplying with X_t' on both sides and taking the expectation,.

$$E\left[X_{t+1}X_t^{'}\right] = AE\left[X_t X_t^{'}\right] + B\underbrace{E\left[\varepsilon_{t+1}X_t^{'}\right]}_{0}$$

ε_{t+1} and X_t' are independent

$$M_1 = AM_0 + 0$$

$$A = M_1 M_0^{-1}$$

Multivariate Stochastic models

Post multiplying with $X_{t+1}{}'$ on both sides and taking the expectation,.

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

$$X_{t+1}X_{t+1}^{'} = AX_tX_{t+1}^{'} + B\varepsilon_{t+1}X_{t+1}^{'}$$

$$\underbrace{E\left[X_{t+1}X_{t+1}^{'}\right]}_{M_0} = AE\left[X_tX_{t+1}^{'}\right] + BE\left[\varepsilon_{t+1}X_{t+1}^{'}\right]$$

Multivariate Stochastic models

$$M_1 = E\left[X_t X_{t-1}^{'}\right]$$

$$M_1^{'} = \left\{E\left[X_t X_{t-1}^{'}\right]\right\}^{'}$$

$$= E\left[\left\{X_t X_{t-1}^{'}\right\}^{'}\right]$$

$$= E\left[X_{t-1} X_t^{'}\right]$$

or

$$M_1^{'} = E\left[X_t X_{t+1}^{'}\right]$$

$(AB)'$

$B'A'$

Multivariate Stochastic models

$$\varepsilon_{t+1} X_{t+1}^{'} = \varepsilon_{t+1} \left\{ A X_t + B \varepsilon_{t+1} \right\}^{'}$$

$$= \varepsilon_{t+1} X_t^{'} A^{'} + \varepsilon_{t+1} \varepsilon_{t+1}^{'} B^{'}$$

Taking expectation on both sides,

$$E\left[\varepsilon_{t+1} X_{t+1}^{'}\right] = E\left[\varepsilon_{t+1} X_t^{'} A^{'} + \varepsilon_{t+1} \varepsilon_{t+1}^{'} B^{'}\right]$$

$$= E\left[\varepsilon_{t+1} X_t^{'} A^{'}\right] + E\left[\varepsilon_{t+1} \varepsilon_{t+1}^{'} B^{'}\right]$$

$$= 0 + I B^{'}$$

$$= B^{'}$$

Since ε_{t+1} has unit variance

Multivariate Stochastic models

Substituting in the equation,

$$E\left[X_{t+1}X_{t+1}^{'}\right] = AE\left[X_{t}X_{t+1}^{'}\right] + BE\left[\varepsilon_{t+1}X_{t+1}^{'}\right]$$

$$M_0 = AM_1^{'} + BB^{'}$$

$$A = M_1 M_0^{-1}$$

$$M_0 = M_1 M_0^{-1} M_1^{'} + BB^{'}$$

$$BB^{'} = M_0 - M_1 M_0^{-1} M_1^{'}$$

If $C = BB'$

$$C = M_0 - M_1 M_0^{-1} M_1^{'}$$

Multivariate Stochastic models

- B does not have a unique solution.
- One method is to assume B to be a lower triangular matrix.

$$BB' = \begin{bmatrix} b(1,1) & 0 & 0 & . & . & 0 \\ b(2,1) & b(2,2) & 0 & . & . & 0 \\ . & . & & & & \\ . & . & & & & \\ b(p,1) & b(p,2) & . & . & . & b(p,p) \end{bmatrix} \begin{bmatrix} b(1,1) & b(2,1) & . & . & . & b(p,1) \\ 0 & b(2,2) & . & . & . & b(p,2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & b(p,p) \end{bmatrix}$$

$$C = \begin{bmatrix} c(1,1) & c(1,2) & c(1,3) & . & . & c(1,p) \\ c(2,1) & c(2,2) & c(2,3) & . & . & c(2,p) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ c(p,1) & c(p,2) & . & . & . & c(p,p) \end{bmatrix}$$

Multivariate Stochastic models

- The diagonal elements of the B matrix are obtained as,

$$b(1,1) = c(1,1)^{1/2}$$

$$b(2,2) = \left\{c(2,2) - b^2(2,1)\right\}^{1/2}$$

.

.

.

$$b(k,k) = \left\{c(k,k) - b^2(k,k-1) - b^2(k,k-2) - \ldots - b^2(k,1)\right\}^{1/2}$$

These elements are obtained one by one, using also the expressions for the k^{th} row elements given in the next slide

Multivariate Stochastic models

- The elements in the k^{th} row are obtained as,

$$b(k,1) = \frac{c(k,1)}{b(1,1)}$$

$$b(k,2) = \frac{c(k,2) - b(2,1)b(k,1)}{b(2,2)}$$

.

.

$$b(k,j) = \frac{c(k,j) - b(j,1)b(k,1) - b(j,2)b(k,2) \ldots\ldots b(j,j-1)b(k,j-1)}{b(j,j)}$$

Example – 1

The annual flow in MCM at two sites P and Q is given below. Generate the first two values of data from these two sites.

Year	1	2	3	4	5	6	7	8	9	10
Annual flow at site P (MCM)	4946	7017	6653	6355	5908	5327	4548	3556	3852	5319
Annual flow at site Q (MCM)	5142	6240	5648	5977	6008	5045	4630	4604	4250	6182

Year	11	12	13	14	15	16	17	18	19
Annual flow at site P (MCM)	4631	5746	5111	5419	6060	7336	3736	3780	6034
Annual flow at site Q (MCM)	4703	6582	5461	5288	5440	7546	4634	4823	5577

Example – 1 (Contd.)

Site	P	Q
Mean	5333	5462
Std.dev.	1125.1	823.5

M_0 matrix is cross correlation matrix of lag zero

$$M_0 = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cc} P & \quad\quad\quad Q \end{array} \\ \begin{bmatrix} r_{P,P}(0) & r_{P,Q}(0) \\ r_{Q,P}(0) & r_{Q,Q}(0) \end{bmatrix} \end{array}$$

Example – 1 (Contd.)

$$r_{P,Q}\left(0\right) = \frac{\sum_{i=1}^{n}\left(x_{P,i} - \overline{x}_P\right)\left(x_{Q,i} - \overline{x}_Q\right)}{\left(n\right)s_P s_Q}$$

$$M_0 = \begin{bmatrix} 1 & 0.796 \\ 0.796 & 1 \end{bmatrix}$$

M_1 matrix is cross correlation matrix of lag one

$$M_1 = \begin{matrix} & P & Q \\ P & r_{P,P}\left(1\right) & r_{P,Q}\left(1\right) \\ Q & r_{Q,P}\left(1\right) & r_{Q,Q}\left(1\right) \end{matrix}$$

Example – 1 (Contd.)

$$r_{P,Q}\left(1\right) = \frac{\sum_{i=1}^{n}\left(x_{P,i} - \overline{x}_P\right)\left(x_{Q,i+1} - \overline{x}_Q\right)}{\left(n-1\right)s_P s_Q}$$

$$M_1 = \begin{bmatrix} 0.302 & 0.164 \\ 0.02 & -0.118 \end{bmatrix}$$

$$A = M_1 M_0^{-1}$$

$$M_0^{-1} = \begin{bmatrix} 2.73 & -2.17 \\ -2.17 & 2.73 \end{bmatrix}$$

Example – 1 (Contd.)

$$A = M_1 M_0^{-1}$$

$$= \begin{bmatrix} 0.302 & 0.164 \\ 0.02 & -0.118 \end{bmatrix} \begin{bmatrix} 2.73 & -2.17 \\ -2.17 & 2.73 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix}$$

Example – 1 (Contd.)

$$C = M_0 - M_1 M_0^{-1} M_1'$$

$$M_0 = \begin{bmatrix} 1 & 0.796 \\ 0.796 & 1 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 0.302 & 0.164 \\ 0.02 & -0.118 \end{bmatrix}$$

$$M_0^{-1} = \begin{bmatrix} 2.73 & -2.17 \\ -2.17 & 2.73 \end{bmatrix} \qquad M_1' = \begin{bmatrix} 0.302 & 0.02 \\ 0.164 & -0.118 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.89 & 0.76 \\ 0.76 & 0.95 \end{bmatrix}$$

Example – 1 (Contd.)

$$c(1,1) = 0.89, c(1,2) = c(2,1) = 0.76,$$

$$c(2,2) = 0.95$$

$$b(1,1) = c(1,1)^{1/2} = (0.89)^{1/2} = 0.94$$

$$b(2,1) = \frac{c(2,1)}{b(1,1)} = \frac{0.76}{0.94} = 0.81$$

$$b(k,1) = \frac{c(k,1)}{b(1,1)}$$

$$b(2,2) = \left\{c(2,2) - b^2(2,1)\right\}^{1/2}$$

$$= \left\{0.95 - 0.81^2\right\}^{1/2} = 0.54$$

Example – 1 (Contd.)

$$b(1,1) = 0.94,\ b(2,1) = 0.81,\ b(2,2) = 0.54$$

$$B = \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix}$$

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

$$\begin{bmatrix} x_{P,t+1} \\ x_{Q,t+1} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} x_{P,t} \\ x_{Q,t} \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} \varepsilon_{P,t+1} \\ \varepsilon_{Q,t+1} \end{bmatrix}$$

Example – 1 (Contd.)

$$\begin{bmatrix} x_{P,t+1} \\ x_{Q,t+1} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} x_{P,t} \\ x_{Q,t} \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} \varepsilon_{P,t+1} \\ \varepsilon_{Q,t+1} \end{bmatrix}$$

The initial value of $x_{P,0}$ and $x_{Q,0}$ are considered as zero
$\varepsilon_{P,1}$ = -0.134 and $\varepsilon_{Q,1}$ = -0.268

$$\begin{bmatrix} x_{P,1} \\ x_{Q,1} \end{bmatrix} = \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} -0.134 \\ -0.268 \end{bmatrix}$$

$$= \begin{bmatrix} -0.126 \\ -0.254 \end{bmatrix}$$

Example – 1 (Contd.)

$$\begin{bmatrix} x_{P,2} \\ x_{Q,2} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} x_{P,1} \\ x_{Q,1} \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} \varepsilon_{P,2} \\ \varepsilon_{Q,2} \end{bmatrix}$$

$\varepsilon_{P,2}$ = 1.639 and $\varepsilon_{Q,2}$ = 0.134

$$\begin{bmatrix} x_{P,2} \\ x_{Q,2} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} -0.126 \\ -0.254 \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} 1.639 \\ 0.134 \end{bmatrix}$$

$$= \begin{bmatrix} 1.543 \\ 1.449 \end{bmatrix}$$

Example – 1 (Contd.)

Site	X	Y
Mean	5333	5462
Std.dev.	1125.1	823.5

Generated annual flow at Site X :

$$Q_{X,1} = \bar{x}_X + x(X,1)s_X$$

$Q_{X,1}$ = 5333 – 0.126*1125.1 = 5191.2 MCM

$$Q_{X,2} = \bar{x}_X + x(X,2)s_X$$

$Q_{X,2}$ = 5333 + 1.543*1125.1 = 7069 MCM

Similarly at Site Y :

$Q_{Y,1}$ = 5462 – 0.254*823.5 = 5252.8 MCM
$Q_{Y,2}$ = 5462 + 1.449*823.5 = 6655.3 MCM

CASE STUDIES