INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

Lecture -34

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Summary of the previous lecture

- Multivariate stochastic models
  - Cross correlation

$$r_{j,h}(k) = \frac{\sum_{i=1}^{n} \left(x_{j,i} - \bar{x}_j\right)\left(x_{h,i+k} - \bar{x}_h\right)}{(n-k)\, s_j s_h}$$

  - Two site Markov model

$$X_{h,t} = \bar{x}_h + r_{j,h}(0)\frac{s_h}{s_j}\left(X_{j,t} - \bar{x}_j\right) + u_t s_h \sqrt{1 - r_{j,h}^2(0)}$$

Multivariate Stochastic models

Multisite Markov model:

- Multisite generation requires simultaneous generation of data at several sites while preserving the correlation between the data at various sites.
- Consider $x_{j,t}$

$$x_{j,t} = \frac{\left(x_{j,t}^{d} - \overline{x}_{j} \right)}{s_{j}}$$

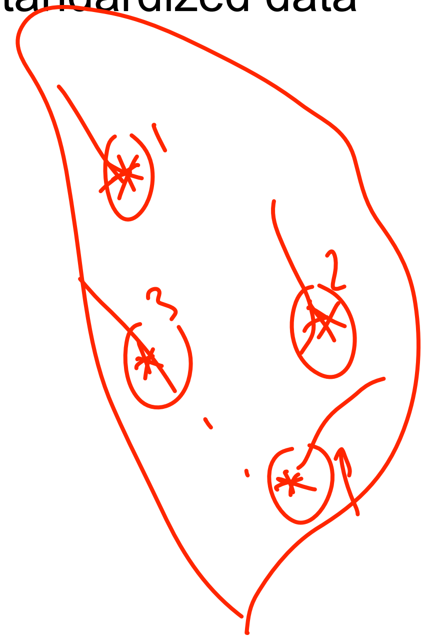
Multivariate Stochastic models

$$X_{t+1} = \mu_x + \rho_1 \left(X_t - \mu_x \right) + u_{t+1} \sigma_x \sqrt{1 - \rho_1^2}$$

- The first order Markov model for site h is

$$x_{h,t+1} = \rho_h \left(1\right) x_{h,t} + \varepsilon_{h,t+1} \sqrt{1 - \rho_h^2 \left(1\right)}$$

μ= 0 and σ =1 because it is standardized data



- The first order Markov model for site j is

$$x_{j,t+1} = \rho_j \left(1\right) x_{j,t} + \varepsilon_{j,t+1} \sqrt{1 - \rho_j^2 \left(1\right)}$$

- The equations are written in matrix form

Multivariate Stochastic models

$$X_{t+1} = EX_t + G\varepsilon$$

where

X_t is a p x 1 vector of standardized values of the variable generated at time t,

E is a p x p diagonal matrix whose jth diagonal element is $\rho_j(1)$,

G is a p x p diagonal matrix whose jth diagonal element is $\sqrt{1-\rho_j^2(1)}$

ε is a p x 1 vector of random variates

Multivariate Stochastic models

- ε is defined to preserve the first order serial correlation (auto correlation) of the x_j 's and the lag zero cross-correlation between x_j and x_h .
- ε is made of elements that are $\varepsilon_{j,t+1}$; each $\varepsilon_{j,t+1}$ is independent of $x_{j,t}$; ε_j is N(0,1)
- The cross correlation between ε_j and ε_h is $\rho^*_{j,h}(0)$,

$$\rho^*_{j,h}(0) = \frac{\{1-\rho_j(1)\rho_h(1)\}\rho_{j,h}(0)}{\sqrt{\{1-\rho_j^2(1)\}\{1-\rho_h^2(1)\}}}$$

- $\rho^*_{j,h}(0)$ reproduces the desired $\rho_{j,h}(0)$, which is the lag zero cross correlation between x_j and x_h .

Multivariate Stochastic models

$$\varepsilon = AD_{\lambda}^{1/2} e$$

where

$D_{\lambda}^{1/2}$ is a p x p diagonal matrix whose j^{th} diagonal element is the square root of the j^{th} largest eigenvalue of the p x p correlation matrix whose elements are $\rho^*_{j,h}(0)$

A is a p x p matrix consisting of eigenvectors of correlation matrix,

e is p x 1 vector of independent observations from N (0,1)

Multivariate Stochastic models

- Matalas (1967) has given a multisite normal generation model that preserves the mean, variance, lag one serial correlation, lag one cross-correlation and lag zero cross-correlation.

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

where

X_t and X_{t+1} are p x 1 vectors representing standardized data corresponding to p sites at time steps t and $t+1$ resp.

Assumption is that the model is multivariate normal.

Ref.: Matalas, N.C. (1967) Mathematical assessment of synthetic hydrology, Water Resources Research 3(4):937-945

Multivariate Stochastic models

ε_{t+1} is N(0,1); p x 1 vector with ε_{t+1} independent of X_t.

A and B are coefficient matrices of size p x p. B is assumed to be lower triangular matrix

$$X_{t+1} = \begin{bmatrix} x(1,t+1) \\ x(2,t+1) \\ . \\ . \\ x(i,t+1) \\ . \\ . \\ x(p,t+1) \end{bmatrix} \qquad X_t = \begin{bmatrix} x(1,t) \\ x(2,t) \\ . \\ . \\ x(i,t) \\ . \\ . \\ x(p,t) \end{bmatrix} \qquad \varepsilon_{t+1} = \begin{bmatrix} \varepsilon(1,t+1) \\ \varepsilon(2,t+1) \\ . \\ . \\ \varepsilon(i,t+1) \\ . \\ . \\ \varepsilon(p,t+1) \end{bmatrix}$$

Multivariate Stochastic models

- The scalar form is

$$x_{i,t+1} = \sum_{j=1}^{p} a_{i,j} x(j,t) + \sum_{j=1}^{i} b_{i,j} \varepsilon(i,t+1)$$

where

$a_{i,j}$ and $b_{i,j}$ denote the (i , j)th elements of the matrices A and B.

Multivariate Stochastic models

Coefficient matrices A and B:

- The expectation of $X_t X_t'$ is denoted by M_0

$$M_0 = E\left[X_t X_t'\right]$$

If $m_0(i, j)$ is a element of M_0 matrix (size p x p) in the i^{th} row and j^{th} column,

$$m_0\left(i, j\right) = E\left[x\left(i,t\right)x\left(j,t\right)\right]$$

Expected value of a matrix is matrix of expected values of individual elements

$$m_0\left(i, j\right) = \frac{1}{n}\sum_{t=1}^{n} x\left(i,t\right)x\left(j,t\right)$$

n is no. of time periods

Jayarami Reddy, P. Stochastic Hydrology, Laxmi publications, New Delhi, 1997

Multivariate Stochastic models

$$m_0(i,j) = \frac{1}{n}\sum_{t=1}^{n}\left(\frac{Q_{i,t} - \bar{Q}_i}{s_i}\right)\left(\frac{Q_{j,t} - \bar{Q}_j}{s_j}\right)$$

Q is the original random variable before standardization e.g., stream flow

i.e., $m_0(i\ ,j)$ is correlation coefficient between the data at sites i and j at time t.

Therefore M_0 is the cross-correlation matrix of lag zero

Multivariate Stochastic models

- The expectation of $X_t X_{t-1}'$ is denoted by M_1

$$M_1 = E\left[X_t X_{t-1}'\right]$$

If $m_1(i\ ,j)$ is a element of M_1 matrix (size p x p) in the i^{th} row and j^{th} column,

$$m_1(i,j) = E\left[x(i,t)x(j,t-1)\right]$$

Expected value of a matrix is matrix of expected values of individual elements

$$m_1(i,j) = \frac{1}{n-1}\sum_{t=2}^{n} x(i,t)x(j,t-1)$$

n is no. of time periods

Multivariate Stochastic models

$$m_1(i,j) = \frac{1}{n-1}\sum_{t=2}^{n}\left(\frac{Q_{i,t}-\bar{Q}_i}{s_i}\right)\left(\frac{Q_{j,t-1}-\bar{Q}_j}{s_j}\right)$$

Q is the original random variable before standardization e.g., stream flow

i.e., $m_1(i\ ,j)$ represents lag one cross correlation between the data at sites i and j .

Therefore M_1 is the cross-correlation matrix of lag one.

Multivariate Stochastic models

Considering the model,

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

Post multiplying with $X_t^{'}$ on both sides and taking the expectation,.

$$E\left[X_{t+1}X_t^{'}\right] = AE\left[X_tX_t^{'}\right] + B\underbrace{E\left[\varepsilon_{t+1}X_t^{'}\right]}_{0}$$

$$M_1 = AM_0 + 0$$

$$A = M_1M_0^{-1}$$

Multivariate Stochastic models

Post multiplying with X_{t+1}' on both sides and taking the expectation,.

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

$$\underbrace{E\left[X_{t+1}X_{t+1}^{'}\right]}_{M_0} = AE\left[X_tX_{t+1}^{'}\right] + BE\left[\varepsilon_{t+1}X_{t+1}^{'}\right]$$

Multivariate Stochastic models

$$M_1 = E\left[X_t X_{t-1}^{'}\right]$$

$$M_1^{'} = \left\{E\left[X_t X_{t-1}^{'}\right]\right\}^{'}$$

$$= E\left[\left\{X_t X_{t-1}^{'}\right\}^{'}\right]$$

$$= E\left[X_{t-1} X_t^{'}\right]$$

or

$$M_1^{'} = E\left[X_t X_{t+1}^{'}\right]$$

Multivariate Stochastic models

$$\varepsilon_{t+1} X'_{t+1} = \varepsilon_{t+1} \left\{ AX_t + B\varepsilon_{t+1} \right\}'$$
$$= \varepsilon_{t+1} X'_t A' + \varepsilon_{t+1} \varepsilon'_{t+1} B'$$

Taking expectation on both sides,

$$E\left[\varepsilon_{t+1} X'_{t+1}\right] = E\left[\varepsilon_{t+1} X'_t A' + \varepsilon_{t+1} \varepsilon'_{t+1} B'\right]$$
$$= E\left[\varepsilon_{t+1} X'_t A'\right] + E\left[\varepsilon_{t+1} \varepsilon'_{t+1} B'\right]$$
$$= 0 + IB'$$
$$= B'$$

Multivariate Stochastic models

Substituting in the equation,

$$E\left[X_{t+1}X_{t+1}^{'}\right] = AE\left[X_{t}X_{t+1}^{'}\right] + BE\left[\varepsilon_{t+1}X_{t+1}^{'}\right]$$

$$M_{0} = AM_{1}^{'} + BB^{'}$$

$$A = M_{1}M_{0}^{-1}$$

$$M_{0} = M_{1}M_{0}^{-1}M_{1}^{'} + BB^{'}$$

$$BB^{'} = M_{0} - M_{1}M_{0}^{-1}M_{1}^{'}$$

If $C = BB'$

$$C = M_{0} - M_{1}M_{0}^{-1}M_{1}^{'}$$

Multivariate Stochastic models

- B does not have a unique solution.
- One method is to assume B is a lower triangular matrix.

$$BB' = \begin{bmatrix} b(1,1) & 0 & 0 & . & . & 0 \\ b(2,1) & b(2,2) & 0 & . & . & 0 \\ . & . & & & & \\ . & . & & & & \\ b(p,1) & b(p,2) & . & . & . & b(p,p) \end{bmatrix} \begin{bmatrix} b(1,1) & b(2,1) & . & . & . & b(p,1) \\ 0 & b(2,2) & . & . & . & b(p,2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & b(p,p) \end{bmatrix}$$

$$C = \begin{bmatrix} c(1,1) & c(1,2) & c(1,3) & . & . & c(1,p) \\ c(2,1) & c(2,2) & c(2,3) & . & . & c(2,p) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ c(p,1) & c(p,2) & . & . & . & c(p,p) \end{bmatrix}$$

Multivariate Stochastic models

- The diagonal elements of the B matrix are obtained as,

$$b(1,1) = c(1,1)^{1/2}$$

$$b(2,2) = \left\{c(2,2) - b^2(2,1)\right\}^{1/2}$$

.

.

.

$$b(k,k) = \left\{c(k,k) - b^2(k,k-1) - b^2(k,k-2) - \ldots - b^2(k,1)\right\}^{1/2}$$

Multivariate Stochastic models

- The elements in the k^{th} row are obtained as,

$$b(k,1) = \frac{c(k,1)}{b(1,1)}$$

$$b(k,2) = \frac{c(k,2) - b(2,1)b(k,1)}{b(2,2)}$$

.

.

$$b(k,j) = \frac{c(k,j) - b(j,1)b(k,1) - b(j,2)b(k,2)......b(j,j-1)b(k,j-1)}{b(j,j)}$$

Multivariate Stochastic models

- Assumption is that the model is multivariate normal.

Example – 1

The annual flow in MCM at two sites X and Y is given below. Generate the first two values of data from these two sites.

Year	1	2	3	4	5	6	7	8	9	10
Annual flow at site X (MCM)	4946	7017	6653	6355	5908	5327	4548	3556	3852	5319
Annual flow at site Y (MCM)	5142	6240	5648	5977	6008	5045	4630	4604	4250	6182

Year	11	12	13	14	15	16	17	18	19
Annual flow at site X (MCM)	4631	5746	5111	5419	6060	7336	3736	3780	6034
Annual flow at site Y (MCM)	4703	6582	5461	5288	5440	7546	4634	4823	5577

Example – 1 (Contd.)

Site	X	Y
Mean	5333	5462
Std.dev.	1125.1	823.5

M_0 matrix is cross covariance matrix of lag zero

$$M_0 = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{bmatrix} r_{X,X}(0) & r_{X,Y}(0) \\ r_{Y,X}(0) & r_{Y,Y}(0) \end{bmatrix} \end{array}$$

Example – 1 (Contd.)

$$r_{X,Y}\left(0\right)=\frac{\sum_{i=1}^{n}\left(x_{X,i}-\bar{x}_{X}\right)\left(x_{Y,i}-\bar{x}_{Y}\right)}{\left(n\right)s_{X}s_{Y}}$$

$$M_0=\begin{bmatrix}1 & 0.796\\ 0.796 & 1\end{bmatrix}$$

M_1 matrix is lag one cross covariance matrix

$$M_1=\begin{matrix} & X & Y\\ X & r_{X,X}\left(1\right) & r_{X,Y}\left(1\right)\\ Y & r_{Y,X}\left(1\right) & r_{Y,Y}\left(1\right)\end{matrix}$$

Example – 1 (Contd.)

$$r_{X,Y}\left(1\right)=\frac{\sum_{i=1}^{n}\left(x_{X,i}-\overline{x}_{X}\right)\left(x_{Y,i+1}-\overline{x}_{Y}\right)}{\left(n-1\right)s_{X}s_{Y}}$$

$$M_1=\begin{bmatrix}0.302 & 0.164\\ 0.02 & -0.118\end{bmatrix}$$

$$A=M_1M_0^{-1}$$

$$M_0^{-1}=\begin{bmatrix}2.73 & -2.17\\ -2.17 & 2.73\end{bmatrix}$$

Example – 1 (Contd.)

$$A = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix}$$

$$C = M_0 - M_1 M_0^{-1} M_1^{'}$$

$$C = \begin{bmatrix} 0.89 & 0.76 \\ 0.76 & 0.95 \end{bmatrix}$$

Example – 1 (Contd.)

$$c(1,1) = 0.89, c(1,2) = c(2,1) = 0.76,$$

$$c(2,2) = 0.95$$

$$b(1,1) = c(1,1)^{1/2} = (0.89)^{1/2} = 0.94$$

$$b(2,1) = \frac{c(2,1)}{b(1,1)} = \frac{0.76}{0.94} = 0.81$$

$$b(2,2) = \left\{c(2,2) - b^2(2,1)\right\}^{1/2}$$

$$= \left\{0.95 - 0.81^2\right\}^{1/2} = 0.54$$

Example – 1 (Contd.)

$$b(1,1) = 0.94,\ b(2,1) = 0.81,\ b(2,2) = 0.54$$

$$B = \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix}$$

$$X_{t+1} = AX_t + B\varepsilon_{t+1}$$

$$\begin{bmatrix} x_{X,t+1} \\ x_{Y,t+1} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} x_{X,t} \\ x_{Y,t} \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} \varepsilon_{X,t+1} \\ \varepsilon_{Y,t+1} \end{bmatrix}$$

Example – 1 (Contd.)

$$\begin{bmatrix} x_{X,1} \\ x_{Y,1} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} x_{X,0} \\ x_{Y,0} \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} \varepsilon_{X,1} \\ \varepsilon_{Y,1} \end{bmatrix}$$

The initial value of $x_{X,0}$ and $x_{Y,0}$ are considered as zero
$\varepsilon_{X,1}$ = -0.134 and $\varepsilon_{Y,1}$ = -0.268

$$\begin{bmatrix} x_{X,1} \\ x_{Y,1} \end{bmatrix} = \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} -0.134 \\ -0.268 \end{bmatrix}$$

$$= \begin{bmatrix} -0.126 \\ -0.254 \end{bmatrix}$$

Example – 1 (Contd.)

$$\begin{bmatrix} x_{X,2} \\ x_{Y,1} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} x_{X,1} \\ x_{Y,1} \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} \varepsilon_{X,2} \\ \varepsilon_{Y,2} \end{bmatrix}$$

$\varepsilon_{X,2}$ = 1.639 and $\varepsilon_{Y,2}$ = 0.134

$$\begin{bmatrix} x_{X,2} \\ x_{Y,1} \end{bmatrix} = \begin{bmatrix} 0.47 & -0.21 \\ 0.31 & -0.37 \end{bmatrix} \begin{bmatrix} -0.126 \\ -0.254 \end{bmatrix} + \begin{bmatrix} 0.94 & 0 \\ 0.81 & 0.54 \end{bmatrix} \begin{bmatrix} 1.639 \\ 0.134 \end{bmatrix}$$

$$= \begin{bmatrix} 1.543 \\ 1.449 \end{bmatrix}$$

Example – 1 (Contd.)

Site	X	Y
Mean	5333	5462
Std.dev.	1125.1	823.5

Generated annual flow at Site X :

$$Q_{X,1} = \bar{x}_X + x(X,1)s_X$$

$Q_{X,1}$ = 5333 – 0.126*1125.1 = 5191.2 MCM

$$Q_{X,2} = \bar{x}_X + x(X,2)s_X$$

$Q_{X,2}$ = 5333 + 1.543*1125.1 = 7069 MCM

Similarly at Site Y :

$Q_{Y,1}$ = 5462 – 0.254*823.5 = 5252.8 MCM
$Q_{Y,2}$ = 5462 + 1.449*823.5 = 6655.3 MCM

CASE STUDIES