INDIAN INSTITUTE OF SCIENCE

# STOCHASTIC HYDROLOGY

Lecture -26

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# Summary of the previous lecture

- Hydrologic data series for frequency analysis
  - Complete duration series
  - Partial duration series
  - Annual exceedence series
  - Extreme value series
- Extreme value distributions
- Frequency factors

frequency factors

$$x_T = \bar{x} + K_T S$$

$$x_T = \mu + K_T \sigma$$

$$p = \frac{1}{T}$$

$$p[X \geq x_T]$$

# Frequency Analysis

Frequency factor for Normal Distribution:

$$x_T = \mu + K_T\sigma$$

$$K_T = \frac{x_T - \mu}{\sigma}$$

$$K_T = w - \frac{2.515517 + 0.802853w + 0.010328w^2}{1 + 1.432788w + 0.189269w^2 + 0.001308w^3}$$

$$w = \left[\ln\left(\frac{1}{p^2}\right)\right]^{1/2} \qquad 0 < p \leq 0.5$$

# Example – 1

Consider the annual maximum discharge Q in cumec, of a river for 45 years :

| Year | Q | Year | Q | Year | Q | Year | Q |
|---|---|---|---|---|---|---|---|
| 1950 | 804 | 1961 | 507 | 1972 | 1651 | 1983 | 1254 |
| 1951 | 1090 | 1962 | 1303 | 1973 | 716 | 1984 | 430 |
| 1952 | 1580 | 1963 | 197 | 1974 | 286 | 1985 | 260 |
| 1953 | 487 | 1964 | 583 | 1975 | 671 | 1986 | 276 |
| 1954 | 719 | 1965 | 377 | 1976 | 3069 | 1987 | 1657 |
| 1955 | 140 | 1966 | 348 | 1977 | 306 | 1988 | 937 |
| 1956 | 1583 | 1967 | 804 | 1978 | 116 | 1989 | 714 |
| 1957 | 1642 | 1968 | 328 | 1979 | 162 | 1990 | 855 |
| 1958 | 1586 | 1969 | 245 | 1980 | 425 | 1991 | 399 |
| 1959 | 218 | 1970 | 140 | 1981 | 1982 | 1992 | 1543 |
| 1960 | 623 | 1971 | 49 | 1982 | 277 | 1993 | 360 |
| | | | | | | 1994 | 348 |

# Example – 1 (Contd.)

Mean, $\bar{x}$ = 756.6 cumec

Standard deviation, s = 639.5 cumec

Determine the frequency factor and obtain the maximum annual discharge value corresponding to 20 year return period using Normal distribution.

# Example – 1 (Contd.)

T = 20

p = 1/20 = 0.05

$$w = \left[\ln\left(\frac{1}{p^2}\right)\right]^{1/2}$$

$$= \left[\ln\left(\frac{1}{0.05^2}\right)\right]^{1/2}$$

$$= 2.45$$

# Example – 1 (Contd.)

$$K_{20} = w - \frac{2.515517 + 0.802853w + 0.01032w^2}{1 + 1.432788w + 0.189269w^2 + 0.001308w^3}$$

$$= 2.45 - \frac{2.515517 + 0.802853 \times 2.45 + 0.01032 \times 2.45^2}{1 + 1.432788 \times 2.45 + 0.189269 \times 2.45^2 + 0.001308 \times 2.45^3}$$

$$= 1.648$$

$$x_{20} = \overline{x} + K_{20}s$$

$$= 756.6 + 1.648 \times 639.5$$

$= 1810.5$ cumec

# Frequency Analysis

Frequency factor for Extreme Value Type I (EV I) Distribution:

$$K_T = -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{T}{T-1}\right)\right]\right\}$$

- To express T in terms of $K_T$,

$$T = \frac{1}{1 - \exp\left\{-\exp\left[-\left(0.5772 + \frac{\pi K_T}{\sqrt{6}}\right)\right]\right\}}$$

Ref: Applied Hydrology by V.T.Chow, D.R.Maidment, L.W.Mays, McGraw-Hill 1998

# Frequency Analysis

$$x_T = \mu + K_T\sigma$$

When $x_T = \mu$ in the equation $K_T = \dfrac{x_T - \mu}{\sigma}$ ; $K_T = 0$

Substituting $K_T = 0$,

$$T = \frac{1}{1 - \exp\left\{-\exp\left[-\left(0.5772 + \dfrac{\pi \times 0}{\sqrt{6}}\right)\right]\right\}}$$

$= 2.33$ years

i.e., the return period of mean of a EV I is 2.33 years

# Example – 2

Consider the annual maximum discharge of a river for 45 years, given in the previous example.

Mean, $\bar{x}$ = 756.6 cumec

Standard deviation, s = 639.5 cumec

Determine the frequency factor and obtain the maximum annual discharge value corresponding to 20 year return period using Extreme Value Type I (EV I) distribution.

# Example – 2 (Contd.)

T = 20 years

$$K_T = -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{T}{T-1}\right)\right]\right\}$$

$$= -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{20}{20-1}\right)\right]\right\}$$

$$= 1.866$$

$$x_T = \bar{x} + K_T s$$

$$= 756.6 + 1.866 \times 639.5$$

$= 1949.9$ cumec

# Frequency Analysis

Frequency factor for Log Pearson Type III Distribution:

The PDF is

$$f(x) = \frac{\lambda^{\beta}(y-\varepsilon)^{\beta-1} e^{-\lambda(y-\varepsilon)}}{x\Gamma(\beta)} \qquad \log x \geq \varepsilon$$

where y = log x

- The data is converted to the logarithmic series by {y} = log {x}.

# Frequency Analysis

- The mean $\bar{y}$ , standard deviation $s_y$, and the coefficient of skewness $C_s$ are calculated for the converted logarithmic series {y}
- The frequency factor for the log Pearson Type III distribution depends on the return period and coefficient of skewness

# Frequency Analysis

- When $C_s$ = 0, the frequency factor is equal to the standard normal deviate z and is calculated as in case of Normal distribution.
- When $C_s \neq 0$, $K_T$ is calculated by (Kite, 1977)

$$K_T = z + \left(z^2 - 1\right)k + \frac{1}{3}\left(z^3 - 6z\right)k^2 - \left(z^2 - 1\right)k^3 + zk^4 + \frac{1}{3}k^5$$

where k = $C_s$ / 6

Ref: Kite, G. W., Frequency and Risk Analysis in Hydrology, Water Resources Publications, Fort Collins, Colorado, 1977

# Example – 3

Consider the annual maximum discharge of a river for 45 years given in the previous example.

Calculate the frequency factor and obtain the maximum annual discharge value  corresponding to 20 year return period using Log Person Type III distribution.

The logarithmic data series is first obtained.

# Example – 3 (Contd.)

Logarithmic values of the data given in the previous example:

| Year | Log Q | Year | Log Q | Year | Log Q | Year | Log Q |
|---|---|---|---|---|---|---|---|
| 1950 | 2.905 | 1961 | 2.705 | 1972 | 3.218 | 1983 | 3.098 |
| 1951 | 3.037 | 1962 | 3.115 | 1973 | 2.855 | 1984 | 2.633 |
| 1952 | 3.199 | 1963 | 2.294 | 1974 | 2.456 | 1985 | 2.415 |
| 1953 | 2.688 | 1964 | 2.766 | 1975 | 2.827 | 1986 | 2.441 |
| 1954 | 2.857 | 1965 | 2.576 | 1976 | 3.487 | 1987 | 3.219 |
| 1955 | 2.146 | 1966 | 2.542 | 1977 | 2.486 | 1988 | 2.972 |
| 1956 | 3.199 | 1967 | 2.905 | 1978 | 2.064 | 1989 | 2.854 |
| 1957 | 3.215 | 1968 | 2.516 | 1979 | 2.210 | 1990 | 2.932 |
| 1958 | 3.200 | 1969 | 2.389 | 1980 | 2.628 | 1991 | 2.601 |
| 1959 | 2.338 | 1970 | 2.146 | 1981 | 3.297 | 1992 | 3.188 |
| 1960 | 2.794 | 1971 | 1.690 | 1982 | 2.442 | 1993 | 2.556 |
| | | | | | | 1994 | 2.542 |

# Example – 3 (Contd.)

The mean, $\bar{y}$ = 2.725 cumec

Standard deviation, s = 0.388 cumec

Coefficient of skewness $C_s$ = -0.2664

T = 20 years

$$w = \left[\ln\left(\frac{1}{p^2}\right)\right]^{1/2}$$

$$= \left[\ln\left(\frac{1}{0.05^2}\right)\right]^{1/2}$$

$$= 2.45$$

# Example – 3 (Contd.)

$$z = w - \frac{2.515517 + 0.802853w + 0.01032w^2}{1 + 1.432788w + 0.189269w^2 + 0.001308w^3}$$

$$= 2.45 - \frac{2.515517 + 0.802853 \times 2.45 + 0.01032 \times 2.45^2}{1 + 1.432788 \times 2.45 + 0.189269 \times 2.45^2 + 0.001308 \times 2.45^3}$$

$$= 1.648$$

$k = C_s / 6$

$= -0.2664/6$

$= -0.0377$

# Example – 3 (Contd.)

$$K_T = 1.648 + \left(1.648^2 - 1\right)\left(-0.0377\right) + \frac{1}{3}\left(1.648^3 - 6\times 1.648\right)\left(-0.0377\right)^2$$
$$-\left(1.648^2 - 1\right)\left(-0.0377\right)^3 + 1.648\times\left(-0.0377\right)^4 + \frac{1}{3}\left(-0.0377\right)^5$$

= 1.581

$$x_T = \overline{x} + K_T s$$
$$= 756.6 + 1.581\times 639.5$$
$$= 1767.6 \text{ cumec}$$

# PROBABILITY PLOTTING

# Probability Plotting

- Probability plotting is a method to check whether a probability distribution fits a set of data or not.
- The data is plotted on specially designed probability paper.
- When the cumulative distribution function (CDF) F (x), is plotted on arithmetic paper versus the value of RV X, usually a straight line does not result.
- To obtain a straight line on arithmetic paper, F(x) would have to be given by expression F(x) = ax+b or f(x) = a; which is an uniform distribution

# Probability Plotting

i.e., if a CDF of a set of data plots a straight line on arithmetic paper, the data follows uniform distribution.

- The probability paper for a given distribution can be developed so that the cumulative distribution plots as a straight line on the paper.

# Probability Plotting

- Constructing probability paper is a process of transforming the arithmetic scale to the probability scale so that the resulting cumulative distribution plot is a straight line

- The plot is prepared with exceedence probability or the return period ‘T’ on abscissa and the magnitude of the event on ordinate.

# Probability Plotting

Construction of Probability paper:

- Mathematical construction.
- Graphical construction

# Probability Plotting

Mathematical construction:

- For some probability distributions, probability paper can be constructed analytically so that the cumulative distribution function plots a straight line, on the paper.
- This can be achieved by transforming the cumulative distribution function to the form

$$Y = aZ + b$$

where Y is a function of parameters and F(x),

Z is a function of parameters and x,

a and b are functions of parameters.

# Probability Plotting

Exponential distribution:

$$F(x) = 1 - e^{-\lambda x}$$

$x > 0 \quad \lambda > 0$

which can be written as,

$$-\ln\{1 - F(x)\} = \lambda x$$

Comparing with $Y = aZ + b$

$a = \lambda$

$$Y = -\ln\{1 - F(x)\}, \; Z = x, \quad a = \lambda \text{ and } b = 0$$

Y is plotted against Z and the corresponding values of F(x) and x are used to label the axes

# Example – 4

Construct probability paper for exponential distribution with $\lambda = 1/3$

Soln:

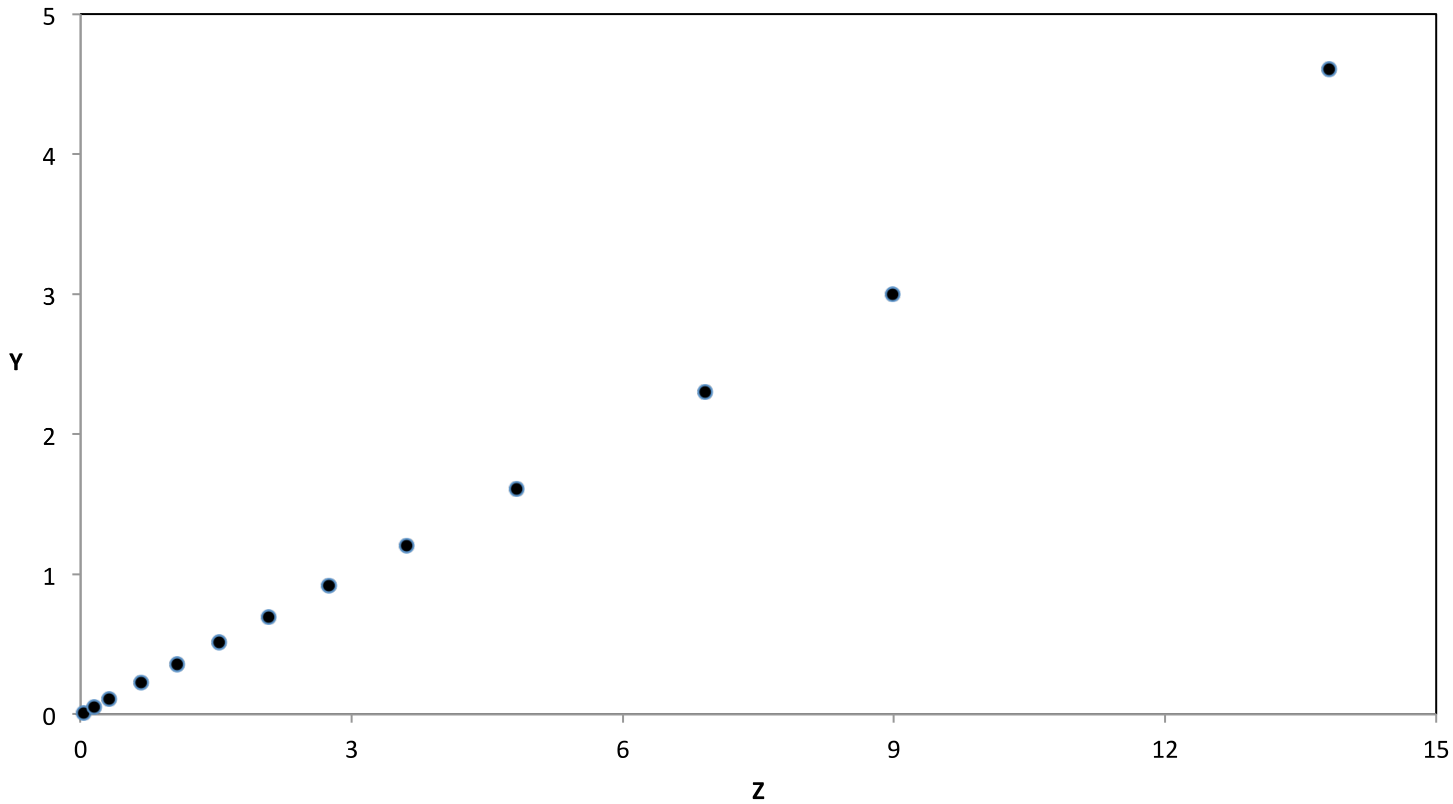
1. The values of F(x) are assumed and corresponding x, Y and Z values are calculated.
2. Y is plotted against Z and the Y axis is labeled with the corresponding value of F(x) and the Z axis with corresponding value of x.

# Example – 4 (Contd.)

| F(x) | Y | x = Z |
|---|---|---|
| 0.01 | 0.010 | 0.030 |
| 0.05 | 0.051 | 0.154 |
| 0.1 | 0.105 | 0.316 |
| 0.2 | 0.223 | 0.669 |
| 0.3 | 0.357 | 1.070 |
| 0.4 | 0.511 | 1.532 |
| 0.5 | 0.693 | 2.079 |
| 0.6 | 0.916 | 2.749 |
| 0.7 | 1.204 | 3.612 |
| 0.8 | 1.609 | 4.828 |
| 0.9 | 2.303 | 6.908 |
| 0.95 | 2.996 | 8.987 |
| 0.99 | 4.605 | 13.816 |

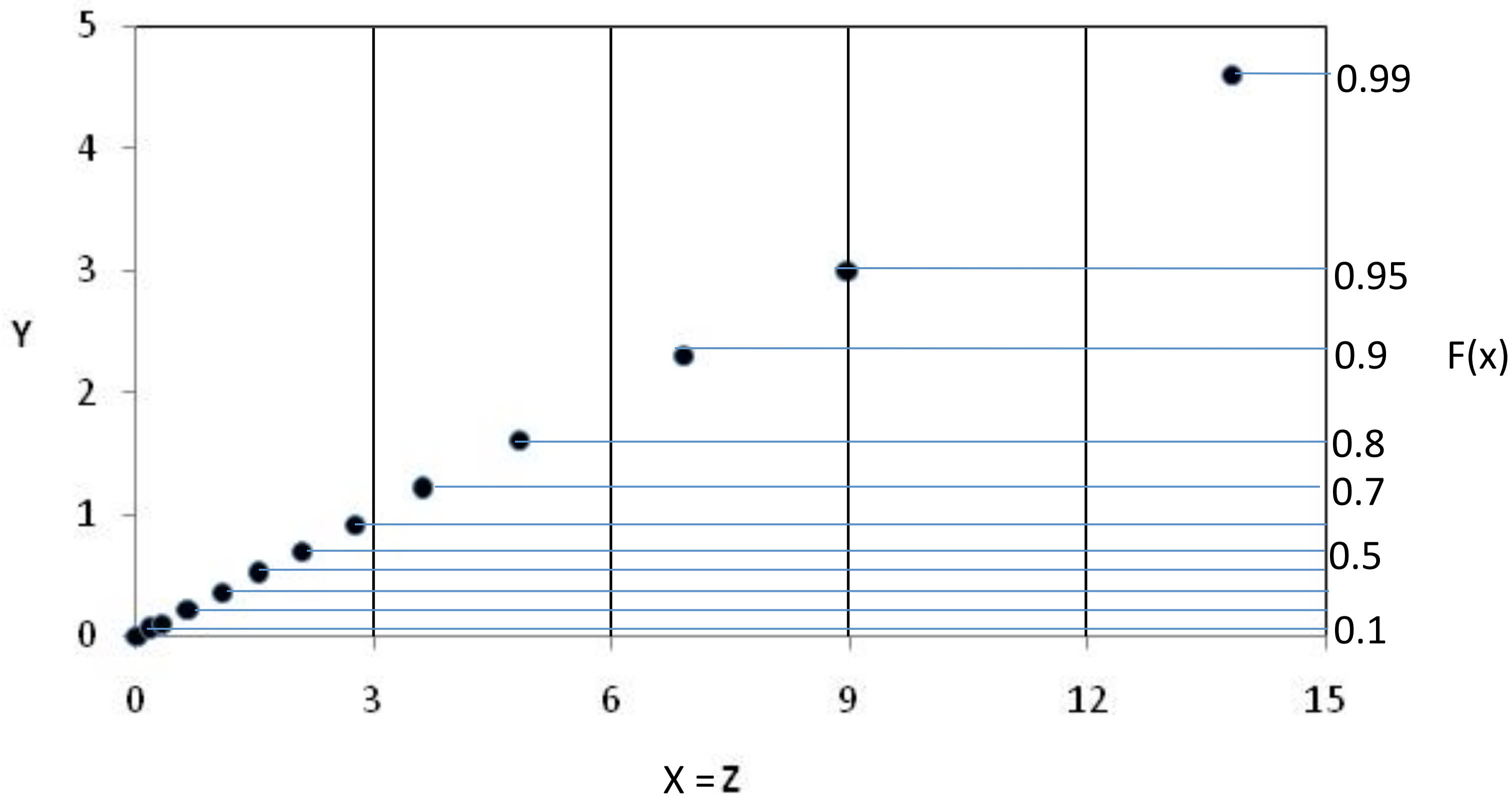
# Example – 4 (Contd.)

Y is plotted against Z :



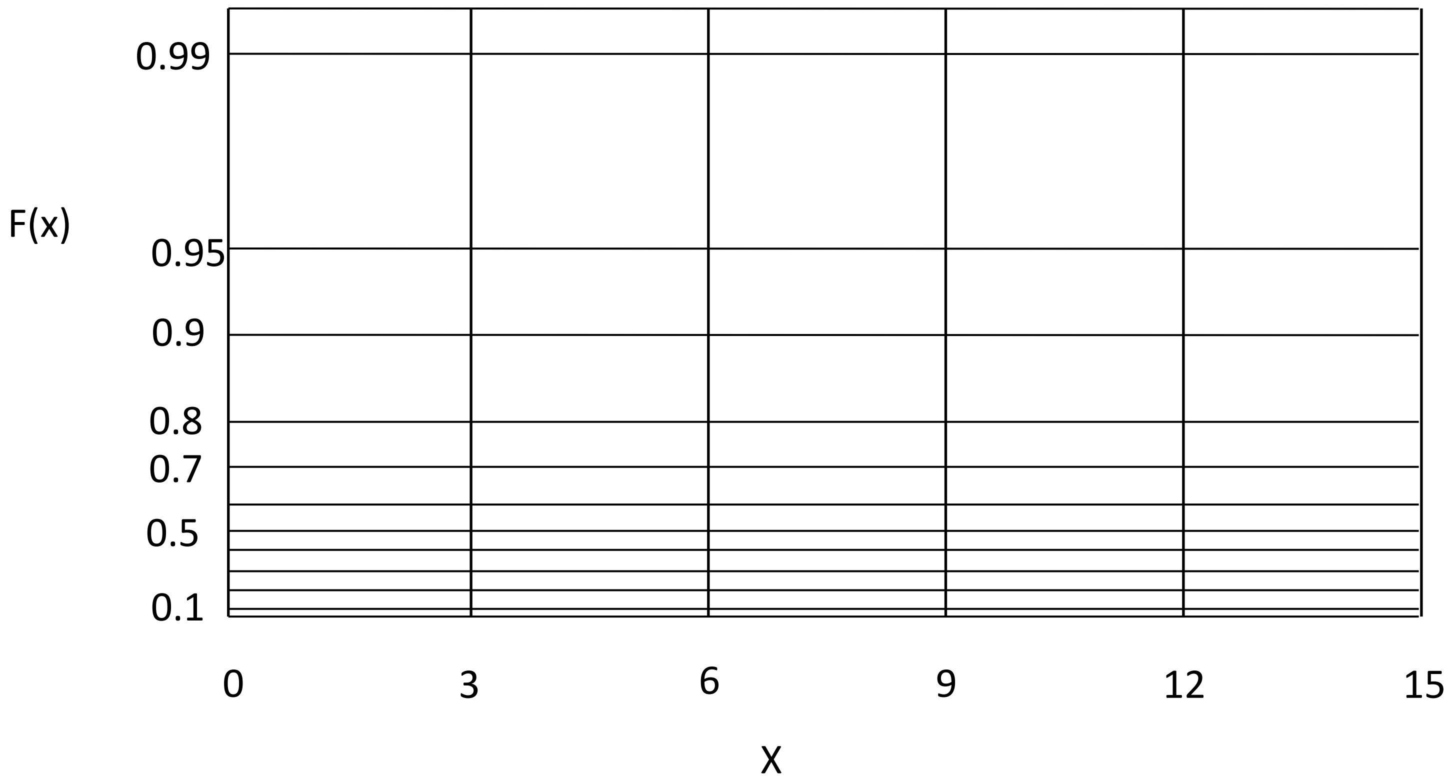
# Example – 4 (Contd.)

Y axis labeled with F(x) and Z with x :



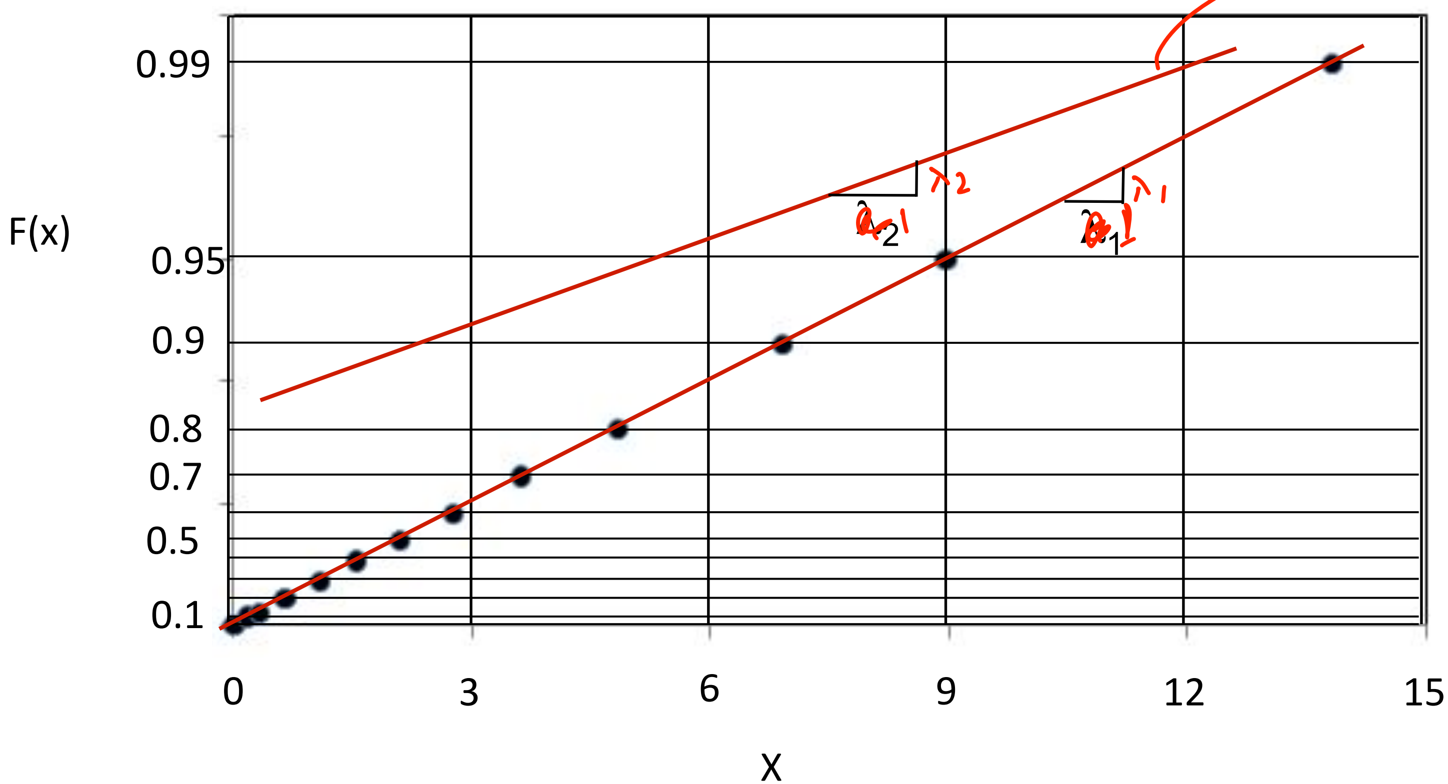
# Example – 4 (Contd.)

Probability paper for exponential distribution:



# Example – 4 (Contd.)

Probability paper for exponential distribution:




Slope of the line gives $\lambda_i$

# Probability Plotting

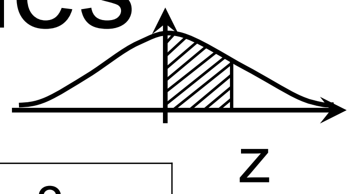
- Any exponential distribution data will plot as a straight line . on the probability paper (exponential)
- The slope of the line will change as $\lambda$ changes.
- Slope of the line gives the $\lambda$ value.
- For many probability distributions, the same graph paper may be used for all values of the parameters of the distribution.
- For some distributions like gamma, a separate graph paper is required for different values of the parameters.
- Many types of probability papers are commercially available.

# Probability Plotting

Graphical construction:

- Graphical construction is done by transforming the arithmetic scale to probability scale so that a straight line is obtained when cumulative distribution function is plotted
- The transformation technique is explained with the normal distribution.
- Consider the coordinates from the standardized normal distribution table.

# Normal Distribution Tables




| z | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| 0 | 0 | 0.008 | 0.016 | 0.0239 | 0.0319 |
| 0.1 | 0.0398 | 0.0478 | 0.0557 | 0.0636 | 0.0714 |
| 0.2 | 0.0793 | 0.0871 | 0.0948 | 0.1026 | 0.1103 |
| 0.3 | 0.1179 | 0.1255 | 0.1331 | 0.1406 | 0.148 |
| 0.4 | 0.1554 | 0.1628 | 0.17 | 0.1772 | 0.1844 |
| 0.5 | 0.1915 | 0.1985 | 0.2054 | 0.2123 | 0.219 |
| 0.6 | 0.2257 | 0.2324 | 0.2389 | 0.2454 | 0.2517 |
| 0.7 | 0.258 | 0.2642 | 0.2704 | 0.2764 | 0.2823 |
| 0.8 | 0.2881 | 0.2939 | 0.2995 | 0.3051 | 0.3106 |
| 0.9 | 0.3159 | 0.3212 | 0.3264 | 0.3315 | 0.3365 |
| 1 | 0.3413 | 0.3461 | 0.3508 | 0.3554 | 0.3599 |

# Probability Plotting

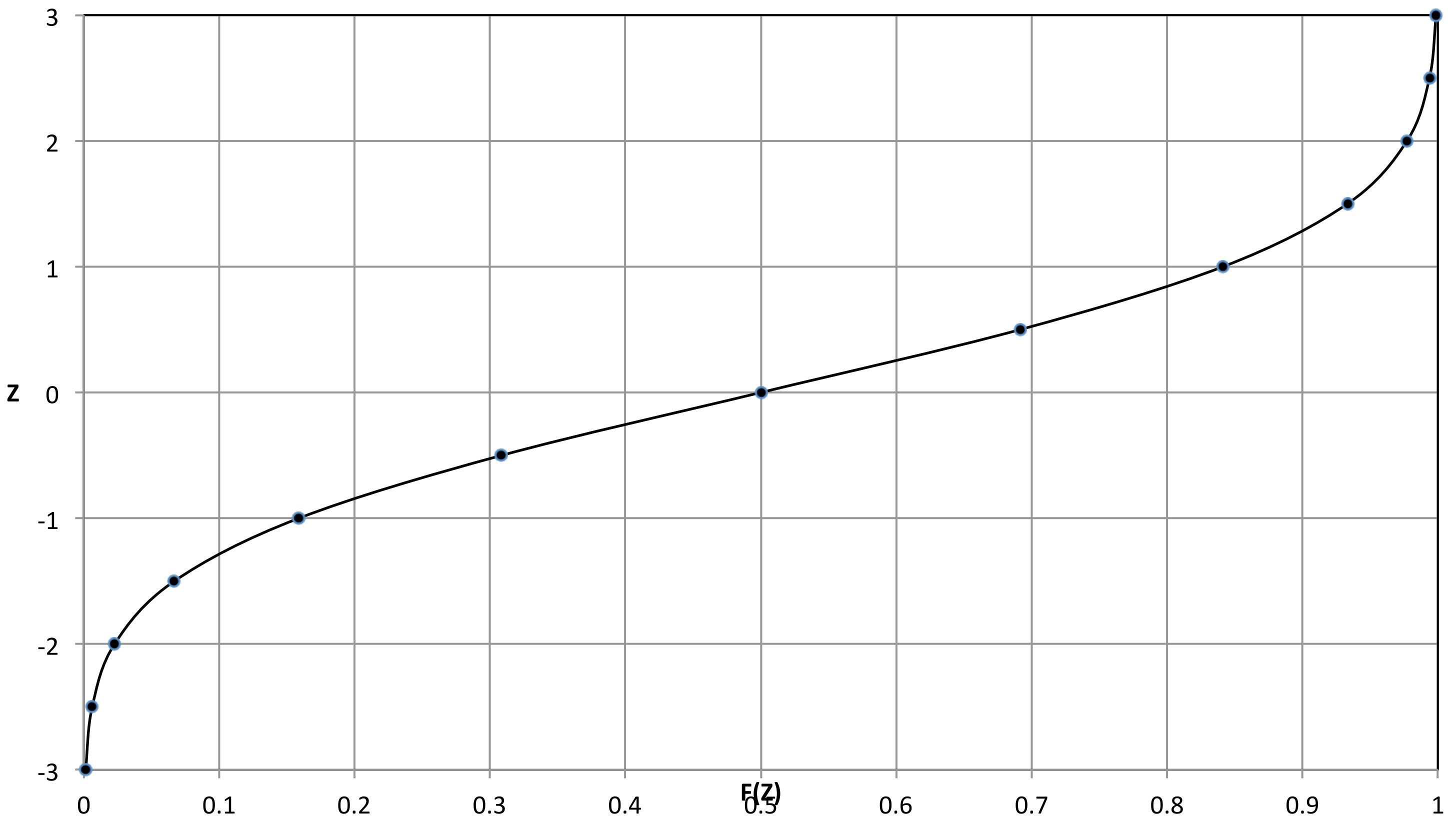
Normal distribution table:

| Z | F(z) |
|---|---|
| -3 | 0.0013 |
| -2.5 | 0.0062 |
| -2 | 0.0227 |
| -1.5 | 0.0668 |
| -1 | 0.1587 |
| -0.5 | 0.3085 |

| Z | F(z) |
|---|---|
| 0 | 0.5 |
| 0.5 | 0.6915 |
| 1 | 0.8413 |
| 1.5 | 0.9332 |
| 2 | 0.9772 |
| 2.5 | 0.9938 |
| 3 | 0.9987 |

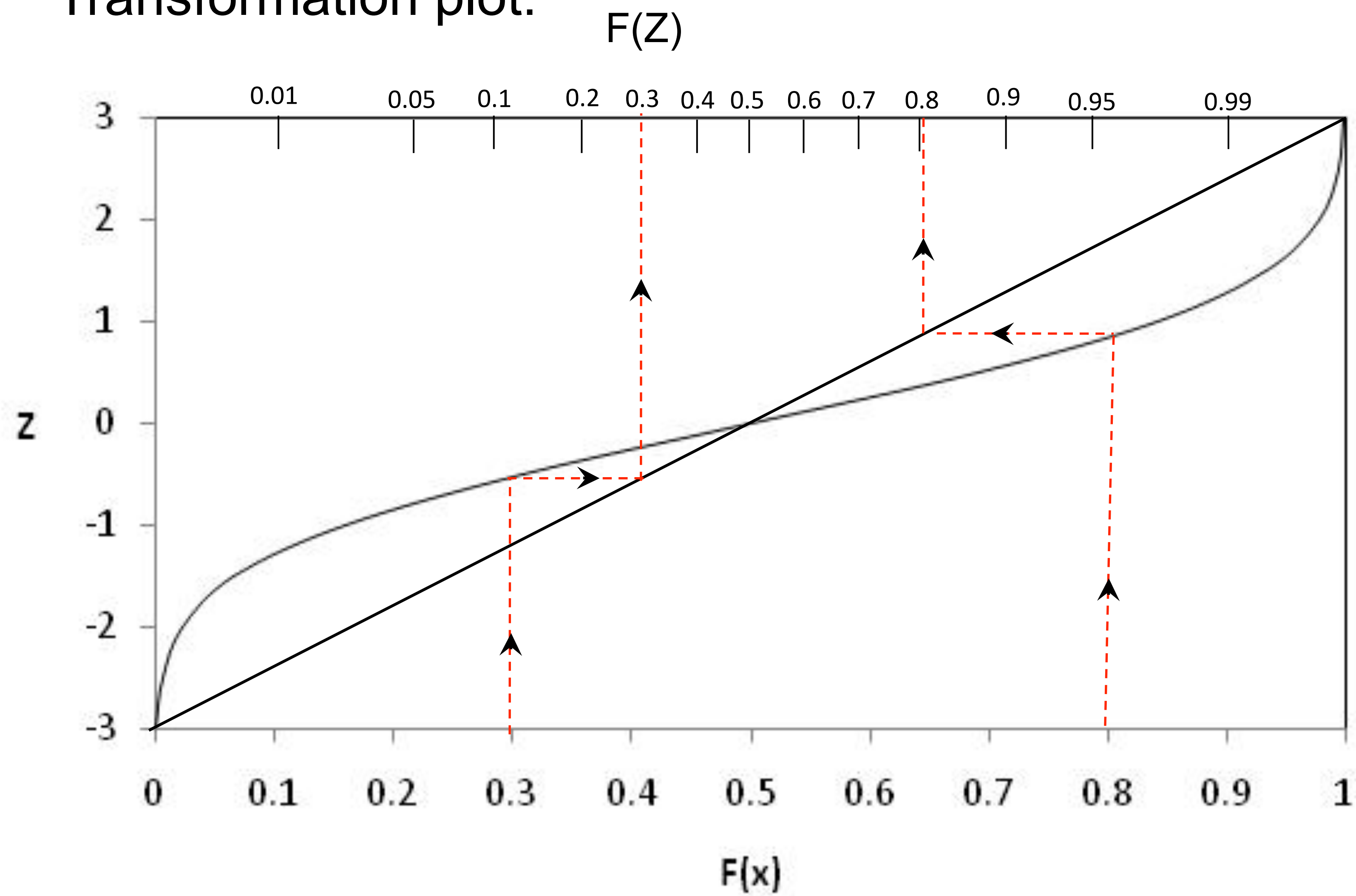
# Probability Plotting

Arithmetic scale plot:



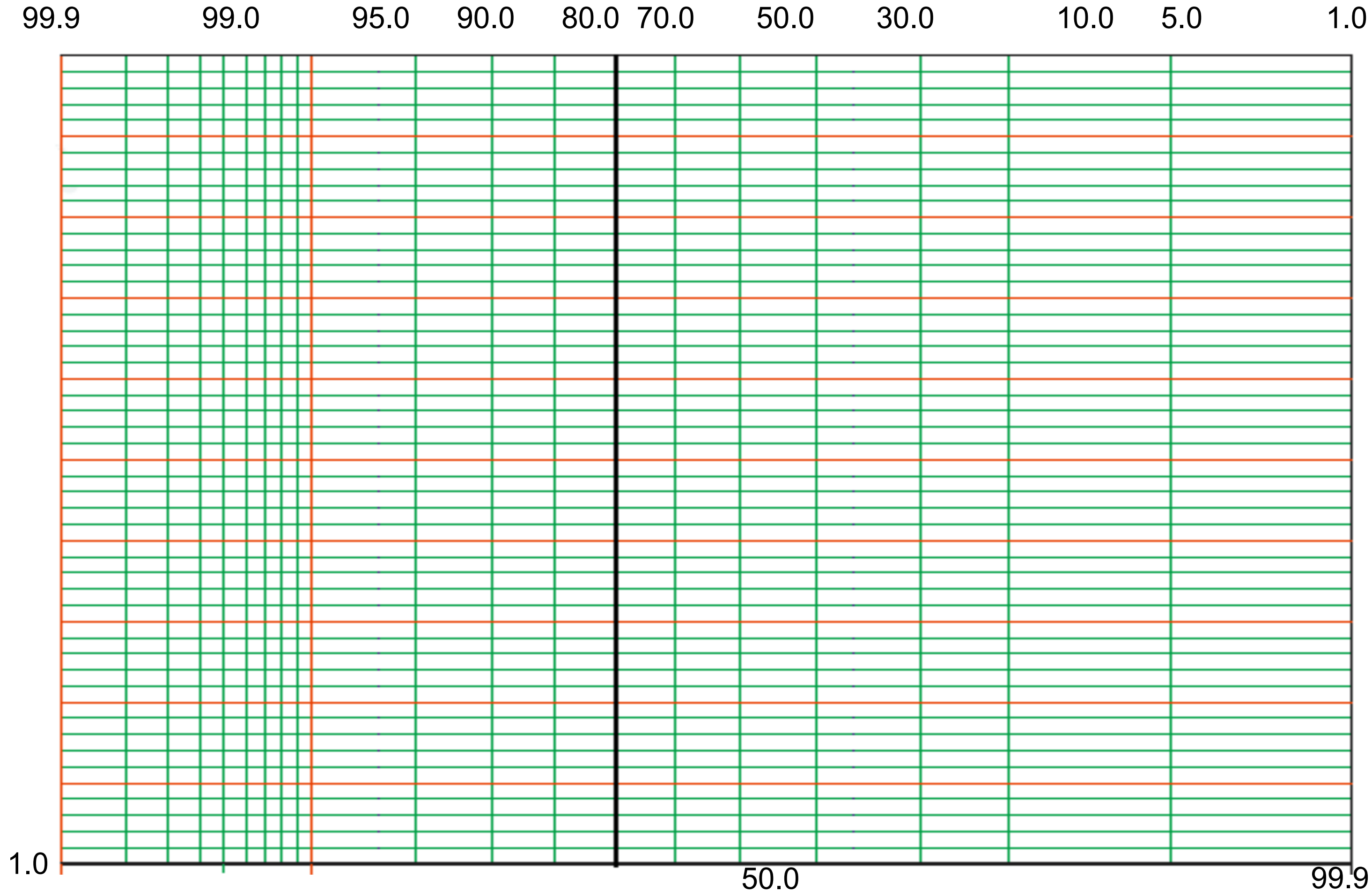
# Probability Plotting

Transformation plot:



# Probability Plotting

Normal probability paper (probabilities in percentage) :




Redrawn from source: http://www.weibull.com/GPaper/

# Probability Plotting

- The purpose of using the probability paper is to linearize the probability relationship
- The plot can be used for interpolation, extrapolation and comparison purposes.
- The plot can also be used for estimating magnitudes with other return periods.
- If the plot is used for extrapolation, the effect of various errors is often magnified.

# Plotting Position

- Plotting position is a simple empirical technique
- Relation between the magnitude of an event verses its probability of exceedence.
- Plotting position refers to the probability value assigned to each of the data to be plotted
- Several empirical methods to determine the plotting positions.
- Arrange the given series of data in descending order
- Assign a order number to each of the data (termed as rank of the data)

# Plotting Position

- First entry as 1, second as 2 etc.
- Let 'n' is the total no. of values to be plotted and 'm' is the rank of a value, the exceedence probability (p) of the $m^{th}$ largest value is obtained by various formulae.
- The return period (T) of the event is calculated by T = 1/p
- Compute T for all the events
- Plot T verses the magnitude of event on semi log or log log paper

# Plotting Position

Formulae for exceedence probability:

California Method:

$$P\left(X \geq x_m\right) = \frac{m}{n}$$

Limitations

– Produces a probability of 100% for m = n

# Plotting Position

Modification to California Method:

$$P\left(X \geq x_m\right) = \frac{m-1}{n}$$

Limitations

- Formula does not produce 100% probability
- If m = 1, probability is zero

# Plotting Position

Hazen's formula:

$$P\left(X \geq x_m\right) = \frac{m - 0.5}{n}$$

Chegodayev's formula:

$$P\left(X \geq x_m\right) = \frac{m - 0.3}{n + 0.4}$$

Widely used in U.S.S.R and Eastern European countries

# Plotting Position

Weibull's formula:

- Most commonly used method
- If 'n' values are distributed uniformly between 0 and 100 percent probability, then there must be n+1 intervals, n–1 between the data points and 2 at the ends.

$$P\left(X \geq x_m\right) = \frac{m}{n+1}$$

- Indicates a return period T one year longer than the period of record for the largest value

# Plotting Position

Most plotting position formulae are represented by:

$$P\left(X \geq x_m\right) = \frac{m-b}{n+1-2b}$$

Where b is a parameter

- E.g., b = 0.5 for Hazen's formula, b = 0.5 for Chegodayev's formula, b = 0 for Weibull's formula
- b = 3/8 0.5 for Blom's formula
- b = 1/3 0.5 for Tukey's formula
- b = 0.44 0.5 for Gringorten's formula

# Plotting Position

- Cunnane (1978) studied the various available plotting position methods based on unbiasedness and minimum variance criteria.
- If large number of equally sized samples are plotted, the average of the plotted points foe each value of m lie on the theoretical distribution line.
- Minimum variance plotting minimizes the variance of the plotted points about the theoretical line.
- Cunnane concluded that the Weibull's formula is biased and plots the largest values of a sample at too small a return period.

# Plotting Position

- For normally distributed data, the best formula is Blom’s plotting position formula (b = 3/8).
- For Extreme Value Type I distribution, the Gringorten formula (b = 0.44) is the best.

# Example – 3

Consider the annual maximum discharge of a river for 45years, plot the data using Weibull's formula

| Year | Data | Year | Data | Year | Data | Year | Data |
|---|---|---|---|---|---|---|---|
| 1950 | 804 | 1961 | 507 | 1972 | 1651 | 1983 | 1254 |
| 1951 | 1090 | 1962 | 1303 | 1973 | 716 | 1984 | 430 |
| 1952 | 1580 | 1963 | 197 | 1974 | 286 | 1985 | 260 |
| 1953 | 487 | 1964 | 583 | 1975 | 671 | 1986 | 276 |
| 1954 | 719 | 1965 | 377 | 1976 | 3069 | 1987 | 1657 |
| 1955 | 140 | 1966 | 348 | 1977 | 306 | 1988 | 937 |
| 1956 | 1583 | 1967 | 804 | 1978 | 116 | 1989 | 714 |
| 1957 | 1642 | 1968 | 328 | 1979 | 162 | 1990 | 855 |
| 1958 | 1586 | 1969 | 245 | 1980 | 425 | 1991 | 399 |
| 1959 | 218 | 1970 | 140 | 1981 | 1982 | 1992 | 1543 |
| 1960 | 623 | 1971 | 49 | 1982 | 277 | 1993 | 360 |
| | | | | | | 1994 | 348 |

# Example – 3 (Contd.)

- The data is arranged in descending order
- Rank  is assigned to the arranged data
- The probability is obtained using

$$P\left(X \geq x_m\right) = \frac{m}{n+1}$$

- Return period is calculated
- The maximum annual discharge verses the return period is plotted

# Example – 3 (Contd.)

| Year | Annual Max. Q | Arranged data | Rank (m) | $P(X \geq x_m)$ | T |
|---|---|---|---|---|---|
| 1950 | 804 | 3069 | 1 | 0.021739 | 46 |
| 1951 | 1090 | 1982 | 2 | 0.043478 | 23 |
| 1952 | 1580 | 1657 | 3 | 0.065217 | 15.33333 |
| 1953 | 487 | 1651 | 4 | 0.086957 | 11.5 |
| 1954 | 719 | 1642 | 5 | 0.108696 | 9.2 |
| 1955 | 140 | 1586 | 6 | 0.130435 | 7.666667 |
| 1956 | 1583 | 1583 | 7 | 0.152174 | 6.571429 |
| 1957 | 1642 | 1580 | 8 | 0.173913 | 5.75 |
| 1958 | 1586 | 1543 | 9 | 0.195652 | 5.111111 |
| 1959 | 218 | 1303 | 10 | 0.217391 | 4.6 |
| 1960 | 623 | 1254 | 11 | 0.23913 | 4.181818 |

# Example – 3 (Contd.)