INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

Lecture -23

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Summary of the previous lecture

- Markov chains
  - Transition probabilities
  - Transition probability matrix (TPM)
  - Steady state Markov chains

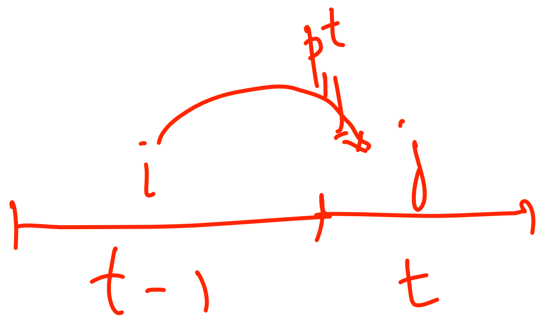
Markov Chains

- stochastic process with the property that value of process X_t at time t depends on its value at time t-1 and not on the sequence of other values

$$P\left[X_t / X_{t-1}, X_{t-2}, \ldots\ldots X_0\right] = P\left[X_t / X_{t-1}\right]$$

$$P\left[X_t = a_j / X_{t-1} = a_i\right] = P_{ij}^t$$

$$p^{(n)} = p^{(0)} \times P^n$$



- At steady state,

$$p = p \times P$$

Example – 1

Consider the TPM for a 2-state first order homogeneous Markov chain as

$$TPM = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

State 1 is a non-rainy day and state 2 is a rainy day

Obtain the

1. probability that day 1 is a non-rainy day given that day 0 is a rainy day
2. probability that day 2 is a rainy day given that day 0 is a non-rainy day
3. probability that day 100 is a rainy day given that day 0 is a non-rainy day

Example – 1 (contd.)

1. probability that day 1 is a non-rainy day given that day 0 is a rainy day

$$TPM = \begin{array}{c} \\ \text{No rain} \\ \text{rain} \end{array} \begin{array}{c} \begin{array}{cc} \text{No rain} & \text{rain} \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

The probability is 0.4

2. probability that day 2 is a rainy day given that day 0 is a non-rainy day

$$p^{(2)} = p^{(1)} \times P$$

$p^{(1)}$, in this case is [0.7 0.3] because it is given that day 0 is a non-rainy day.

Example – 1 (contd.)

$$p^{(2)} = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.61 & 0.39 \end{bmatrix}$$

The probability is 0.39

3. probability that day 100 is a rainy day given that day 0 is a non-rainy day

$$p^{(n)} = p^{(0)} \times P^n$$

Example – 1 (contd.)

$$P^2 = P \times P$$

$$= \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$$P^4 = P^2 \times P^2 = \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{bmatrix}$$

$$P^8 = P^4 \times P^4 = \begin{bmatrix} 0.5715 & 0.4285 \\ 0.5714 & 0.4286 \end{bmatrix}$$

$$P^{16} = P^8 \times P^8 = \begin{bmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{bmatrix}$$

Example – 1 (contd.)

Steady state probability

$$p = \begin{bmatrix} 0.5714 & 0.4286 \end{bmatrix}$$

Verification: at steady state,

$$p = p \times P$$

$$= \begin{bmatrix} 0.5714 & 0.4286 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5714 & 0.4286 \end{bmatrix}$$

Markov Chains

Data generation:

- Data generation from a Markov chain needs knowledge of the initial state and the transitional probability matrix (TPM).
- To determine the state at time 2, get a random number R_u between 0 and 1.
- If this random number R_u is between $\sum_{j=1}^{n-1} p_{ij}$ and $\sum_{j=1}^{n} p_{ij}$

for n = 1, 2, ……m, the next state is taken as state 'n'

Example – 2

Consider a Markov chain model for annual streamflow at a location. Assume the TPM as

$$TPM = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}$$

State 1 – Deficit flow

State 2 – Intermediate Flow

State 3 – Excess flow (Flow above a threshold)

Example – 2 (Contd.)

Consider that the flow is in state 1 at t = 0; generate a sequence of 10 possible flow states corresponding to t = 1, 2, ……..10

The cumulative transition probability matrix (P*) is

$$P^* = \begin{bmatrix} 0.7 & 1 & 1 \\ 0.1 & 0.7 & 1 \\ 0.5 & 0.9 & 1 \end{bmatrix}$$

Example – 2 (Contd.)

given

Time	State at t	R_u	State at t+1
1	1	0.896	2
2	2	0.919	3
3	3	0.682	2
4	2	0.922	3
5	3	0.735	2
6	2	0.435	2
7	2	0.006	1
8	1	0.154	1
9	1	0.549	1
10	1	0.612	1

Example – 3

Consider that last two days weather conditions influence the today's weather condition. i.e., whether or not today is a dry day or a wet day depends on the previous two days weather condition.

Consider,

- If the past two days have been wet days, then it will be a wet day tomorrow with a probability 0.8
- If today is a wet day but yesterday was a dry day, then it will be a wet day tomorrow with probability 0.5
- If yesterday was a wet day but today is a dry day, then tomorrow will be a wet day with probability 0.3

Example – 3 (Contd.)

- If the past two days have been dry days, then tomorrow will be a wet day with probability 0.1

If we let the state at time 't' depend only whether or not it is a wet day at time 't', then the preceding model is not a single step Markov chain

t-1

However, it is possible to transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day.

Example – 3 (Contd.)

i.e., State 1 – If both today and yesterday are wet days

State 2 – If today is a wet day but yesterday is a dry day

State 3 – If yesterday was a wet day but today is a dry day

State 4 – If both today and yesterday are dry days

	t-2	**t-1**	**t**
State 1	W	W	
State 2	D	W	
State 3	W	D	
State 4	D	D	

Example – 3 (Contd.)

- If the past two days have been wet days, then it will be a wet day tomorrow with a probability 0.8

	t-2	t-1
State 1	W	W
State 2	D	W
State 3	W	D
State 4	D	D

t-1	t	p
W	W	0.8
D	W	0
W	D	0.2
D	D	0

$$\begin{bmatrix} 0.8 & 0 & 0.2 & 0 \end{bmatrix}$$

Example – 3 (Contd.)

The preceding would then represent a 4 state Markov chain with transition probability matrix as

$$TPM = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0.1 & 0 & 0.9 \end{bmatrix}$$

Example – 3 (Contd.)

Consider that Monday and Tuesday are wet days, what is the probability that Thursday is a wet day?

Given that Monday and Tuesday are wet days, the probability vector of Wednesday is

$$p^{(W)} = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \end{bmatrix}$$

$$p^{(T)} = p^{(W)} \times P$$

Example – 3 (Contd.)

$$p^{(T)} = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0.1 & 0 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64 & 0.06 & 0.16 & 0.14 \end{bmatrix}$$

Thursday:	Wet	Wet	Dry	Dry

Therefore the probability of Thursday being a wet day = 0.64+0.06 = 0.7

Markov Chains

Difficulties in using Markov chains in hydrology

- Determining the number of states to use.
- Determining the intervals of the variable under study to associate with each state.
- Assigning a number to the magnitude of an event once the state is determined.
- Estimating the large number of parameters involved in even a moderate size Markov chain model.
- Handling situations where some transitions are dependent on several previous time periods while others are dependent on only one prior time period.

Ref: Statistical methods in Hydrology by C.T.Haan, Iowa state university press

FREQUENCY ANALYSIS

Frequency Analysis

- Hydrologic systems are influenced by severe events occurring.
  - e.g., severe storms, floods, droughts.
- The magnitude of an extreme event is inversely proportional to its frequency of occurrence (i.e., severe events occur less frequently).
- Frequency analysis is a procedure for estimating the frequency (or the probability) of occurrence of past and (or) future events

Frequency Analysis

- Objective of frequency analysis of hydrologic data is to relate the magnitude of extreme events to their frequency of occurrence using probability distributions.
- Hydrologic data to be analyzed is assumed to be independent and identically distributed and the hydrologic system is assumed to be stochastic, space-independent and time independent
- The data should be properly selected so that the assumptions of independent and identically distributed are satisfied.

Frequency Analysis

- The assumption of identical distribution or the homogeneity is achieved by selecting the observations from same population
- The assumption of independence is achieved by selecting the annual maximum of the variable being analyzed as the successive observations from year to year will be independent.

Frequency Analysis

- The results of flood frequency analysis can be used in many engineering purposes
  - E.g., flood flow frequency analysis can be used in the design of dams, bridges, culverts, flood controlling devices
  - Urban flooding