INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

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Course Contents

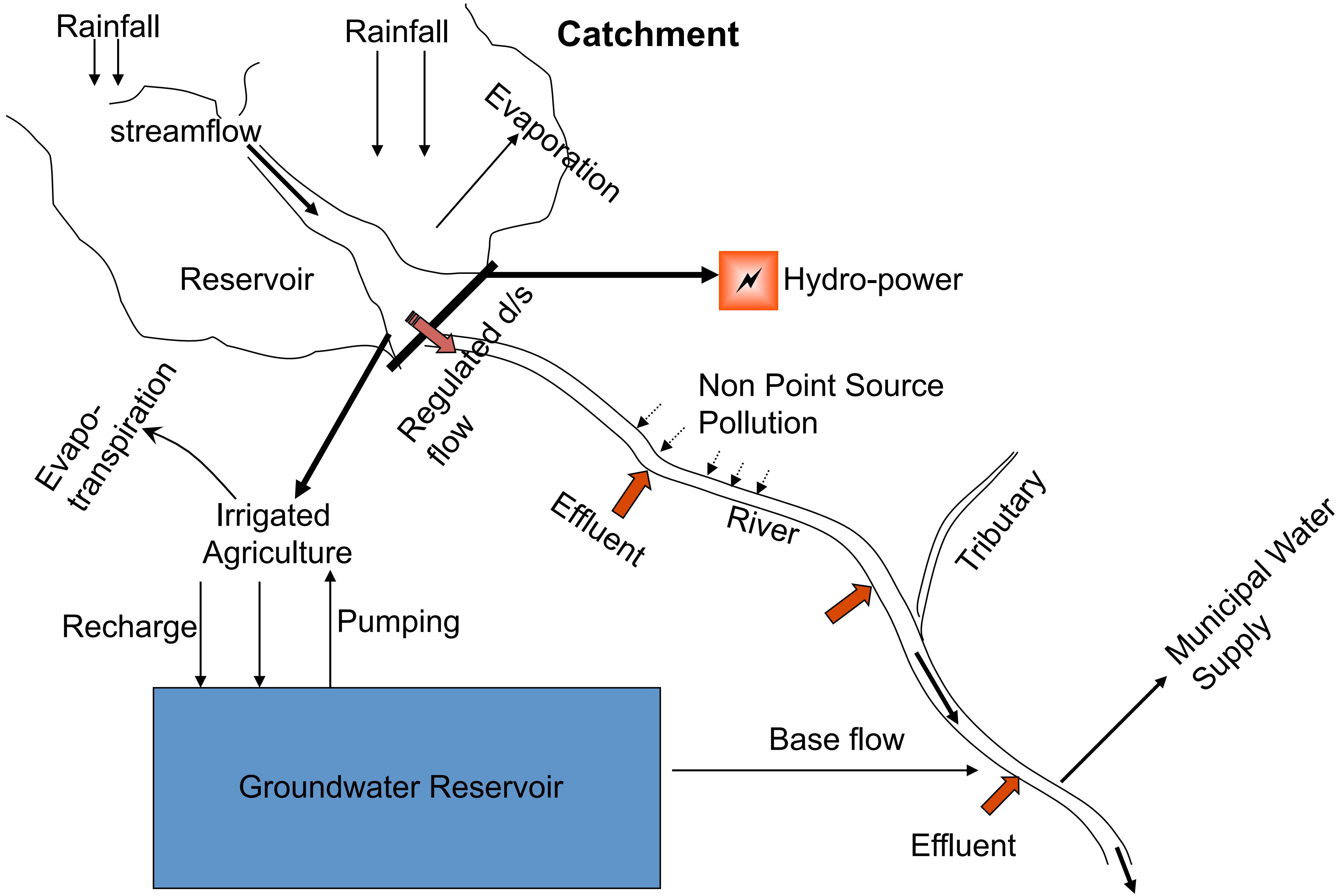
- Introduction to Random Variables (RVs)
- Probability Distributions - One dimensional RVs
- Higher Dimensional RVs – Joint Distribution; Conditional Distribution; Independence
- Properties of Random Variables
- Parameter Estimation – Maximum Likelihood Method and Method of Moments
- Commonly Used Distributions in Hydrology
- Hydrologic Data Generation
- Introduction to Time Series Analysis
- Purely stochastic Models; Markov Processes

Course Contents (contd)

- Analysis in the Frequency Domain : Spectral Density
- Auto Correlation and Partial Auto Correlation
- Auto Regressive Moving Average Models
  - (Box-Jenkins models – model identification; Parameter estimation ; calibration and validation ; Simulation of hydrologic time series ; Applications to Hydrologic Data Generation and Forecasting)

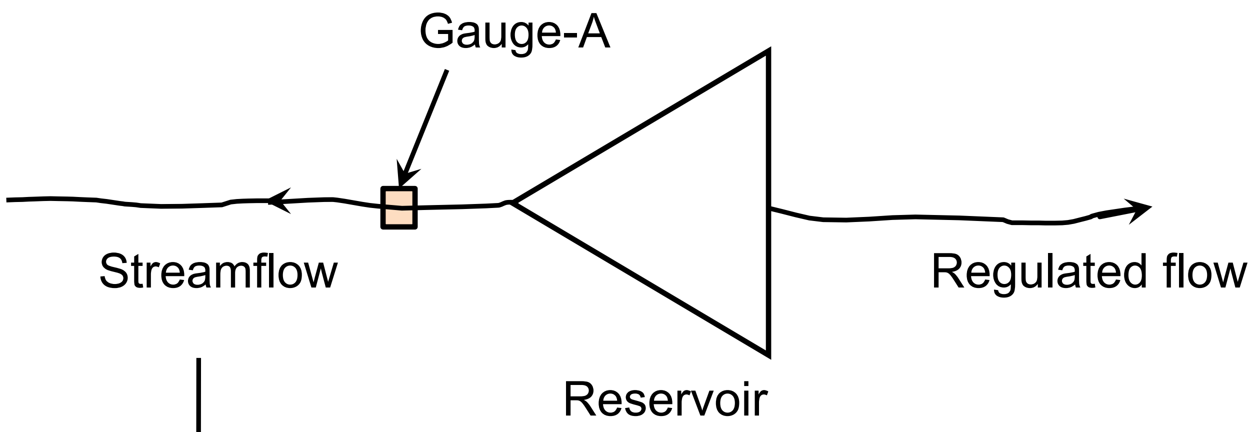
Reference Books

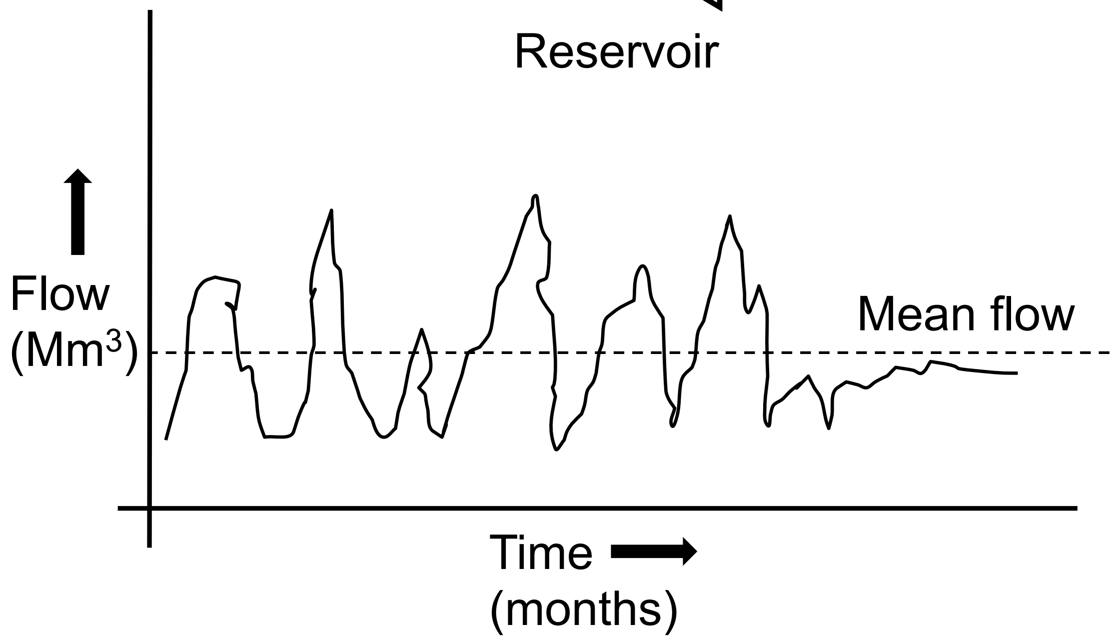
- Haan, C.T., "Statistical Methods in Hydrology", First East-West Press Edition, New Delhi, 1995.
- Bras, R.L. and Rodriguez-Iturbe , “Random Functions and Hydrology”, Dover Publications, New York, USA, 1993.
- Clarke, R.T., "Statistical Models in Hydrology", John Wiley, Chinchester, 1994.
- Yevjevich V. “Probability and statistics in Hydrology”, Water Resources Publications, Colorado, 1972.
- Ang, A.H.S. and Tang, W.H., "Probabilistic concepts in Engineering Planning Design", Vol. 1, Wiley, New York, 1975.



Typical Water Resource System

Stochastic Hydrology - Applications





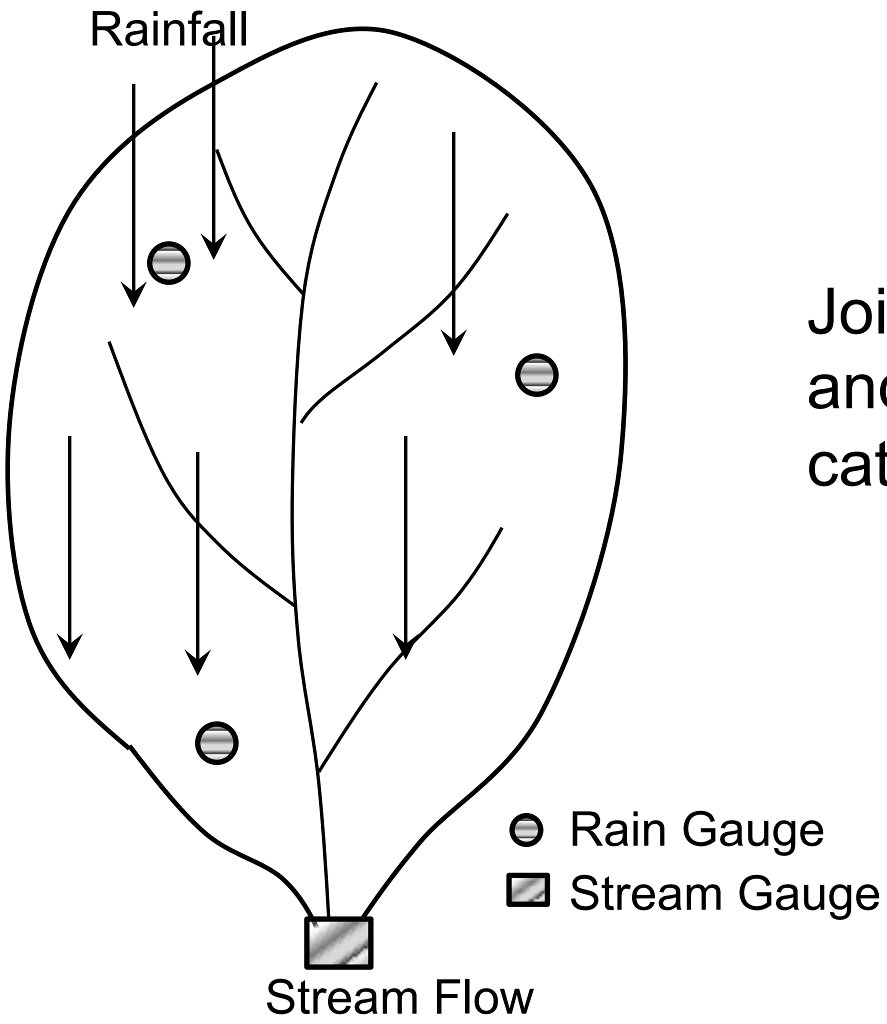
Observed (historical) flows at Gauge - A

History provides a valuable clue to the future

Reservoir Design and Operation

- Medium term forecasts for hydropower/ irrigation/water supply,
- Short term forecasts for flood control

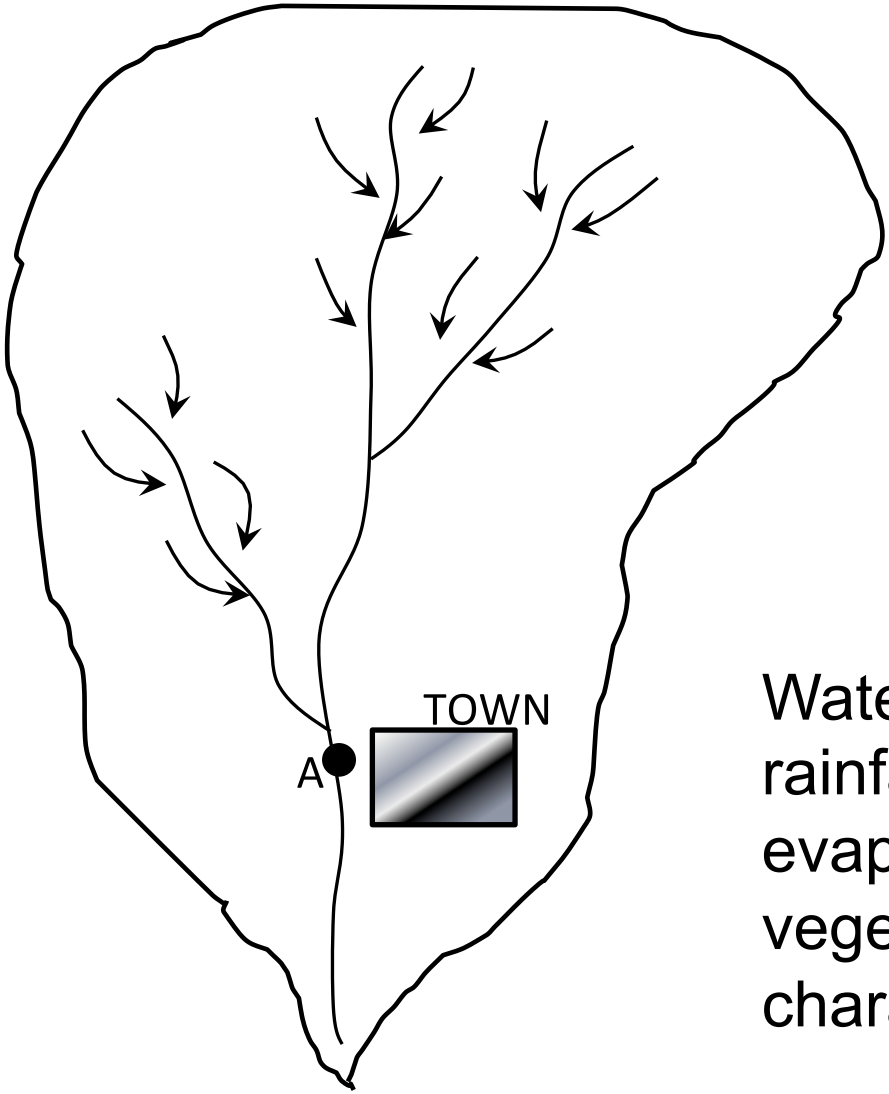
Stochastic Hydrology - Applications



Joint variation of rainfall and streamflow in a catchment

- Rainfall-Runoff relationships

Stochastic Hydrology - Applications



Real-time Flood Forecasting

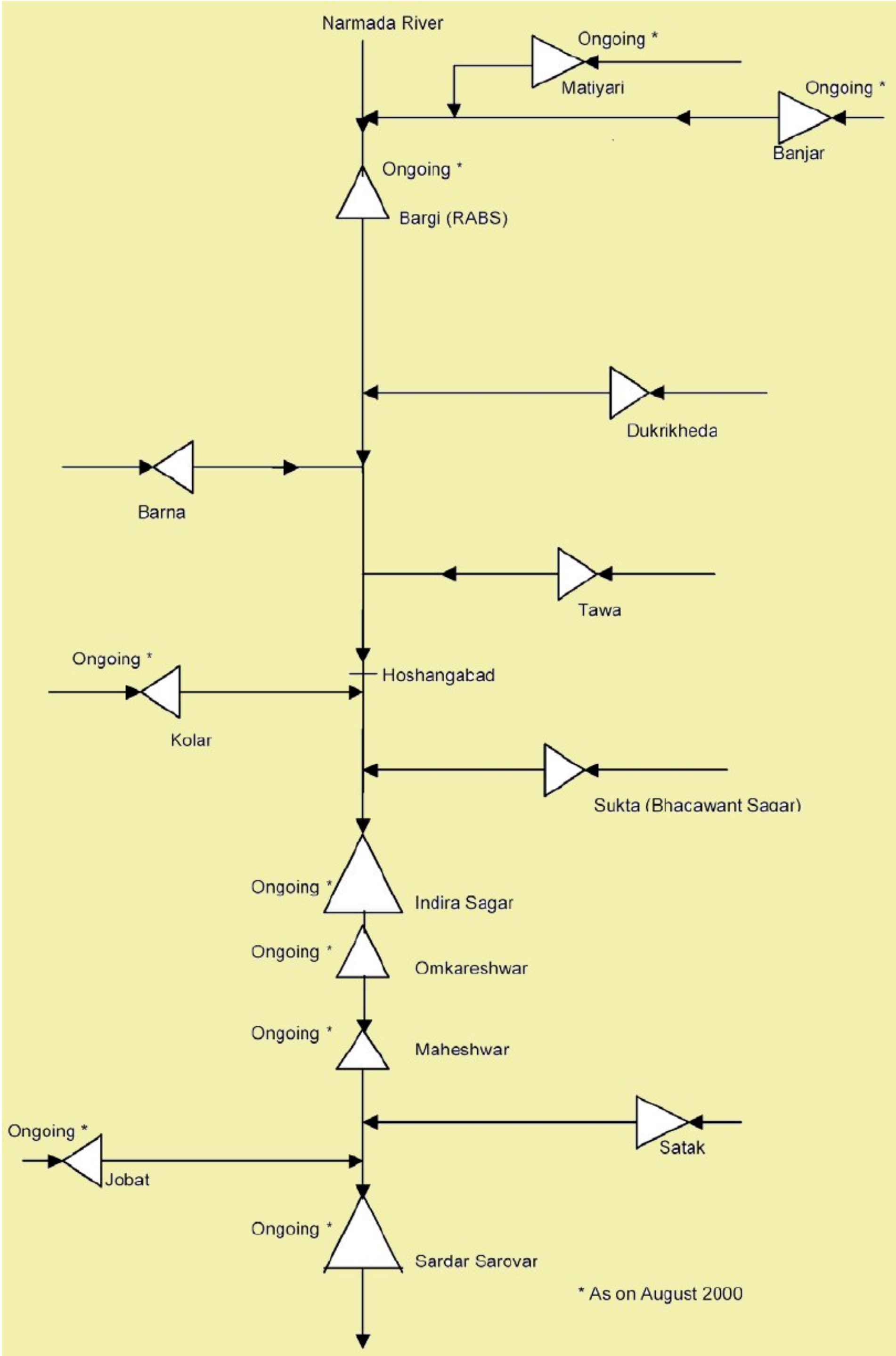
To forecast water levels at ‘A’, with sufficient lead time

Water level at A: Function of rainfall in the catchment upstream, evaporation, infiltration, storage, vegetation and other catchment characteristics.

Stochastic Hydrology - Applications

Multi-reservoir systems

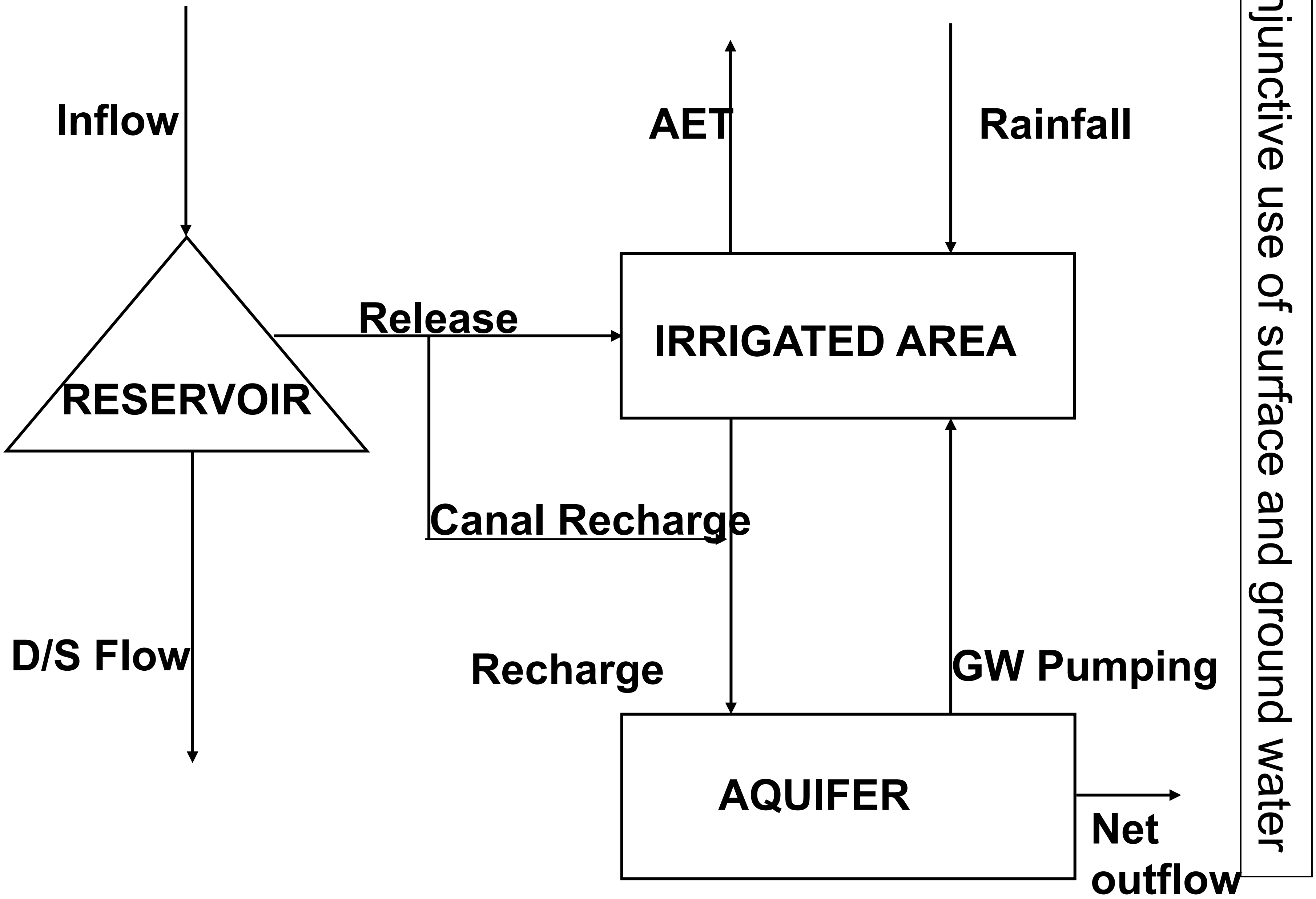
- Flood forecasting
- Intermediate catchment flows
- Long-term operation of the system



Stochastic Hydrology - Applications

- Reliability of Meeting Future Demands
  - How often does the system 'Fail' to deliver?
- Resiliency of the System
  - How quickly can the system recover from failure?
- Vulnerability of the system
  - Effect of a failure (e.g., expected flood damages; deficit hydropower etc.)

Stochastic Hydrology - Applications

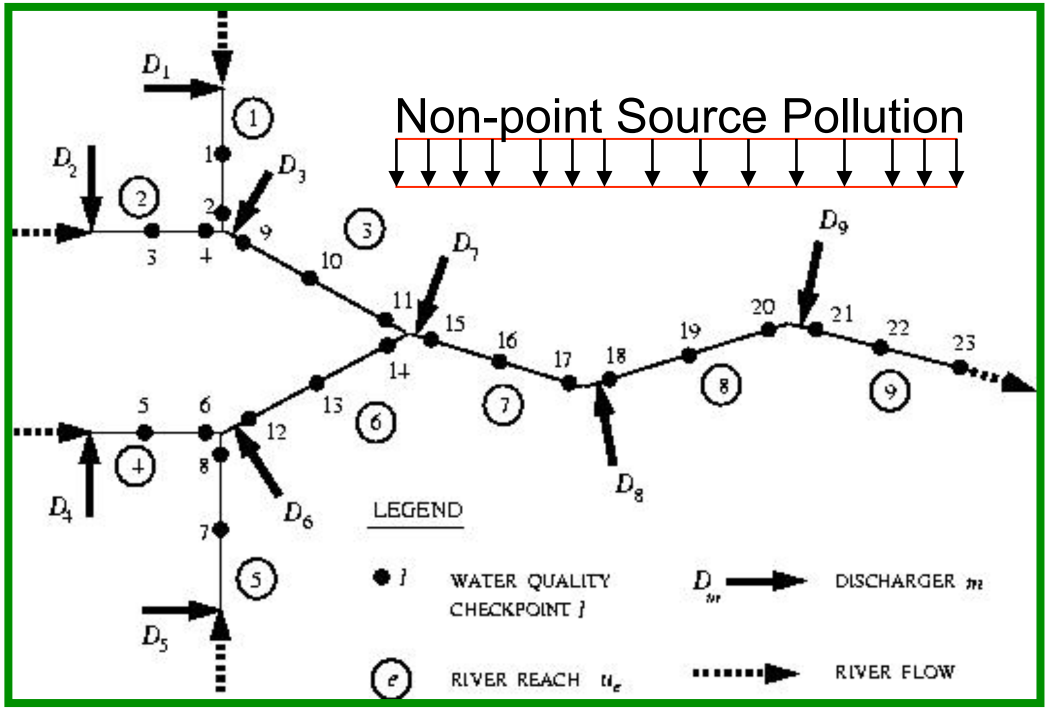


Stochastic Hydrology - Applications

Water Quality in Streams

Governed by :

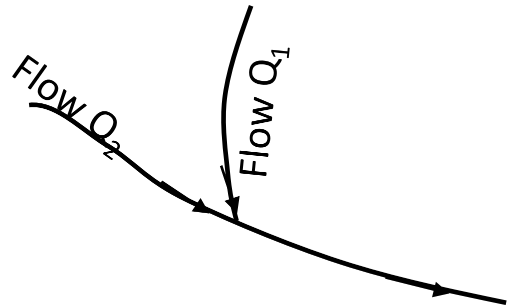
Streamflow, Temperature, Hydraulic properties, Effluent discharges, Non-point source pollution, Reaction rates …..



Stochastic Hydrology - Applications

- Flood frequency analysis
  - – return period of critical events
- Probable Maximum Flood
- Intensity-Duration-Frequency relationships,
- Run-lengths : intervals between rainy days
- Time series, data generation, flow forecasting

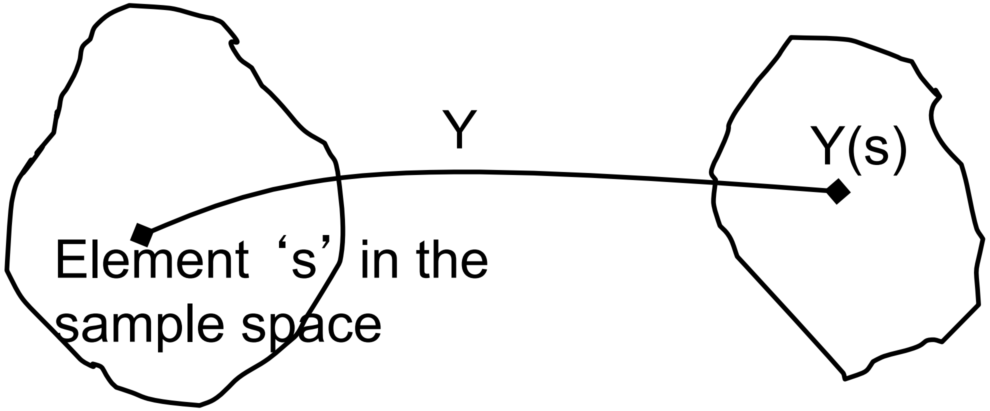
Stochastic Hydrology - Applications



- ☞ Joint variation of flows in two or more streams
- ☞ Urban floods
  - Estimates of design rainfall intensity based on probability concepts
- ☞ Spatial variation in aquifer parameters
- ☞ Uncertainties introduced by climate change
  - Likely changes in frequencies and magnitudes of floods & droughts.
  - Likely changes in stream flow, precipitation patterns, recharge to ground water

Random Variable

Real-valued function defined on the sample space.



Y is a Random Variable

Sample space of an experiment

Possible real values of Y

- Intuitively, a random variable (RV) is a variable whose value cannot be known with certainty, until the RV actually takes on a value; In this sense, most hydrologic variables are random variables

Random Variable

RVs of interest in hydrology

- Rainfall in a given duration
- Streamflow
- Soil hydraulic properties (e.g. permeability, porosity)
- Time between hydrologic events (e.g. floods of a given magnitude)
- Evaporation/Evapotranspiration
- Ground water levels
- Re-aeration rates

Random Variable

- Any function of a random variable is also a random variable.
  - For example, if X is a r.v., then Z = g(X) is also r.v.
- Capital letters will be used for denoting r.v.s and small letters for the values they take

  e.g. X $\longrightarrow$ rainfall, x = 30 mm

  Y $\longrightarrow$ stream flow, y = 300 Cu.m.
- We define 'events' on the r.v. e.g. X=30 ; a $\le$Y $\le$b
- We associate probabilities to occurrence of events
  - represented as P[X=30], P[a$\le$Y$\le$b] etc.

Discrete & Continuous R.V.s

- Discrete R.V.: Set of values a random variable can assume is finite (or countably infinite).
  - No. of rainy days in a month (0,1,2….30)
  - Time (no of years) between two flood events (1,2….)

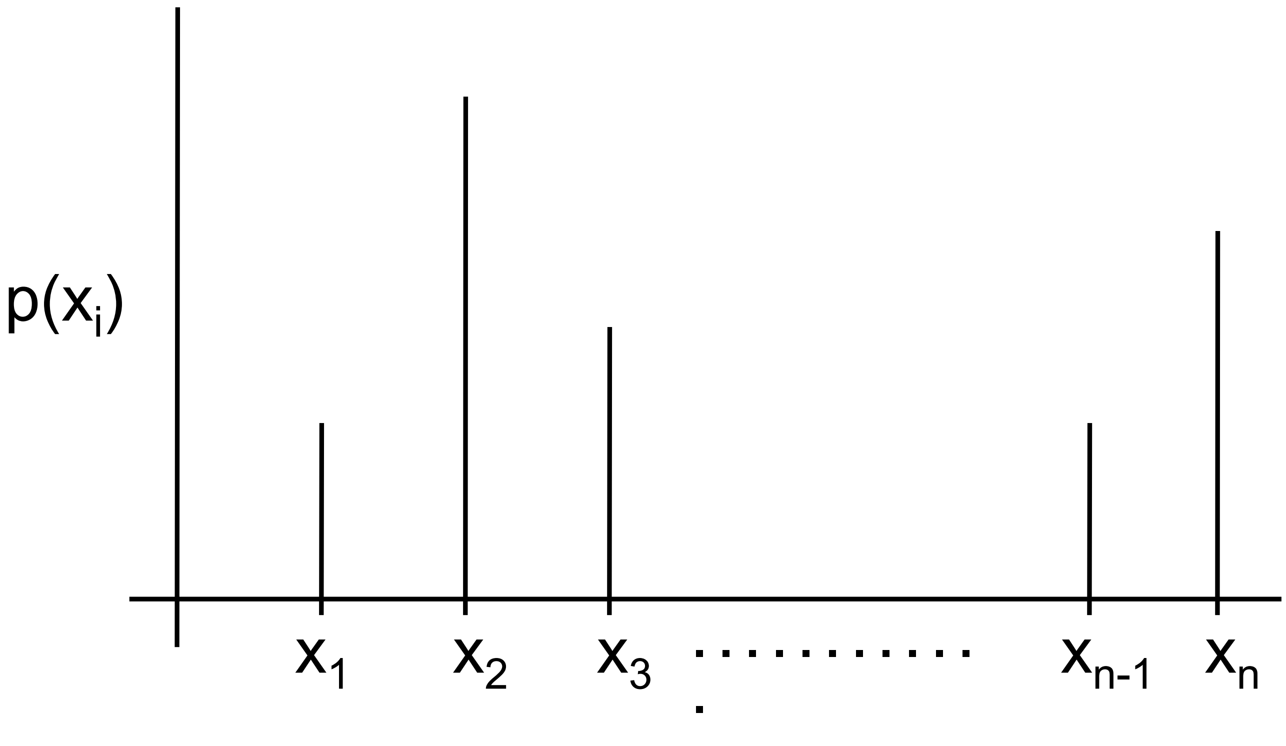
- Continuous R.V.: If the set of values a random variable can assume is infinite (the r.v. can take on values on a continuous scale)
  - Amount of rainfall occurring in a day
  - Streamflow during a period
  - Flood 'peak over threshold'
  - Temperature

Probability Distributions

Discrete random variables: Probability Mass Function

$p(x_i) \geq 0 \quad ; \quad \sum_i p(x_i) = 1$

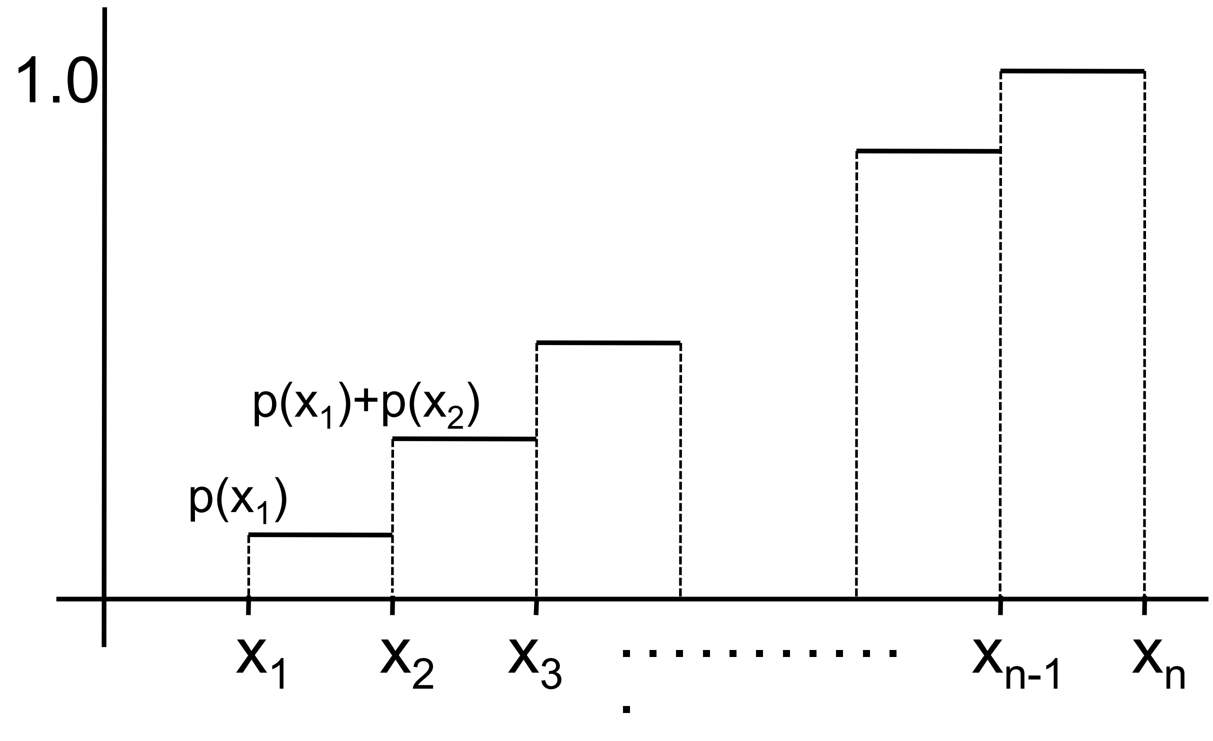
$p(x_i) = P[X = x_i]$



Probability Distributions

Cumulative distribution function : discrete RV

$$F(x) = \sum_{x_i \leq x} p(x_i)$$



- $P[X = x_i] = F(x_i) – F(x_{i-1})$
- The r.v., being discrete, cannot take values other than $x_1, x_2, x_3 \ldots\ldots\ldots x_n$; $P[X = x] = 0$ for $x \neq x_1, x_2, x_3 \ldots\ldots\ldots x_n$
- Some times, it is advantageous to treat continuous r.v.s as discrete rvs.
  – e.g., we may discretise streamflow at a location into a finite no. of class intervals and associate probabilities of the streamflow belonging to a given class

Continuous R.V.s

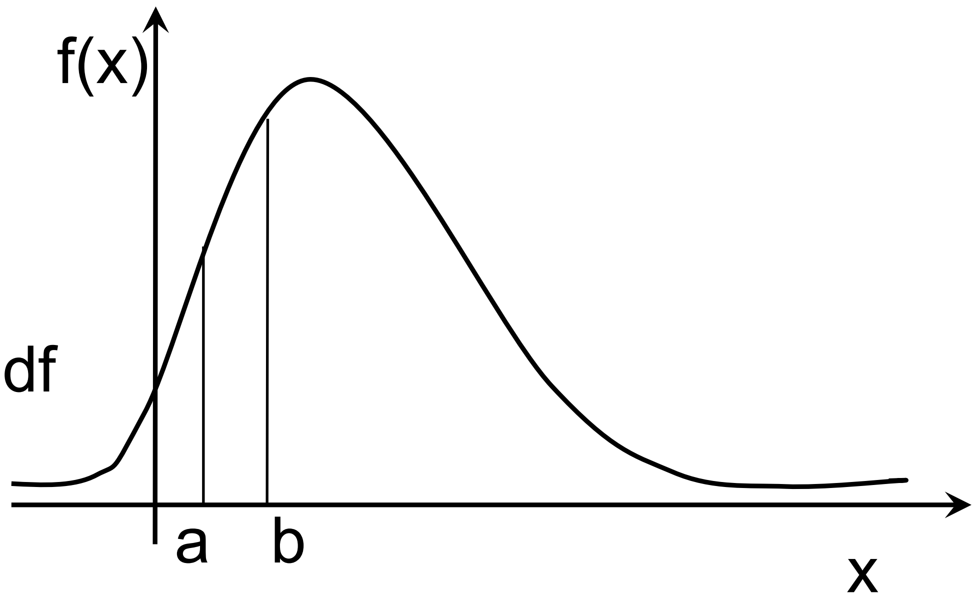
pdf → Probability Density Function f(x)

cdf → Cumulative Distribution Function F(x)

Any function satisfying

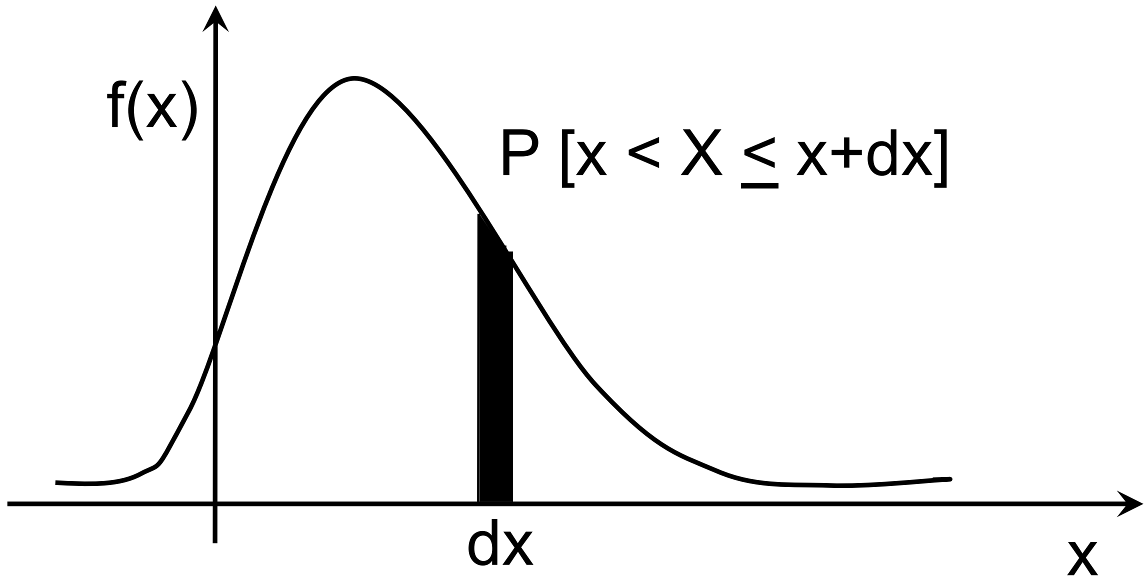
f(x) $\geq$ 0 and

$$\int_{-\infty}^{\infty} f(x) = 1$$ can be a pdf



pdf is NOT probability, but a probability <u>density</u> & therefore pdf value can be more than 1

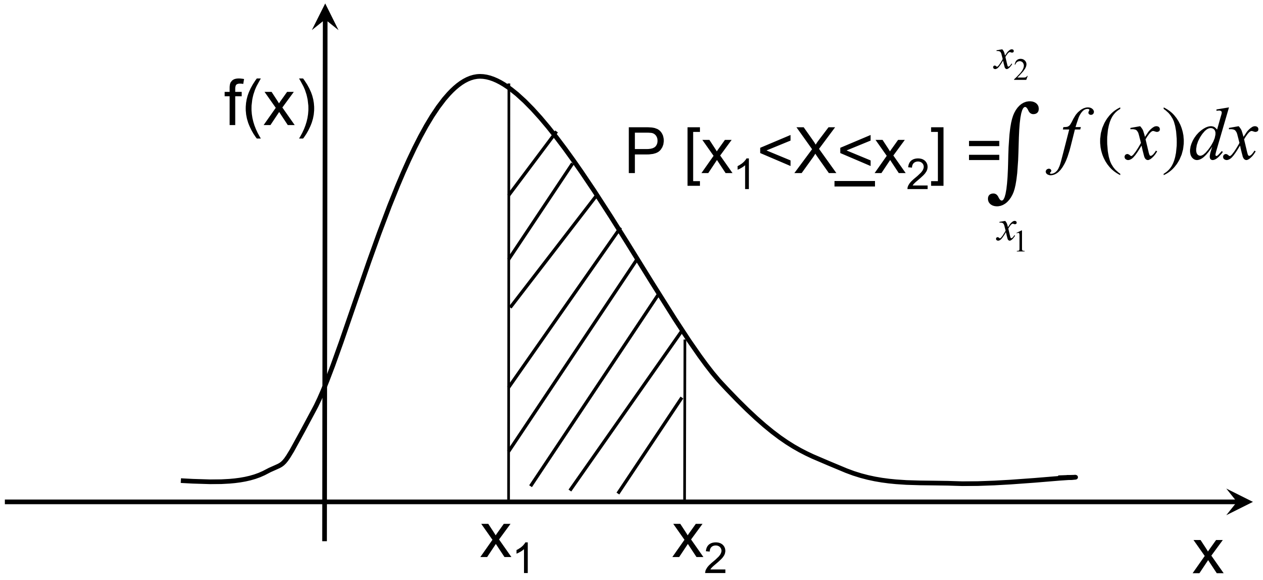
Continuous RVs



$$f(x) = \lim_{dx \to \infty} \frac{\mathrm{P}\left[x < X \leq x + dx\right]}{dx} \quad \text{where} \quad \int_{-\infty}^{\infty} f(x)dx = 1$$

Continuous RVs

PDF → Probability Density Function (Probability mass per unit x)

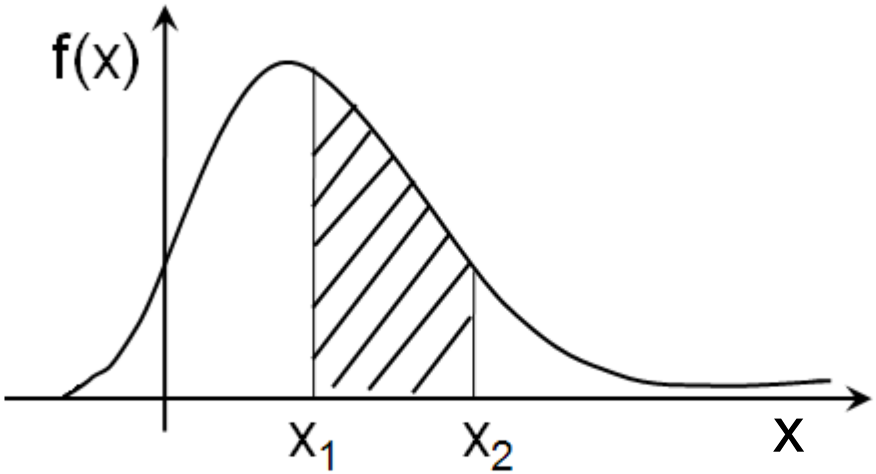


$$P\,[x_1 < X \le x_2] = \int_{x_1}^{x_2} f(x)\,dx$$

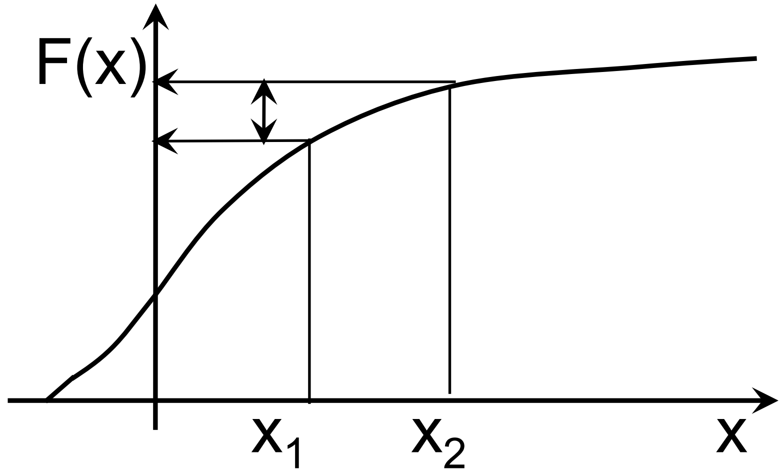
Continuous RVs

$$P[x_1 \le X \le x_2] = \int_{x_1}^{x_2} f(x)\,dx$$

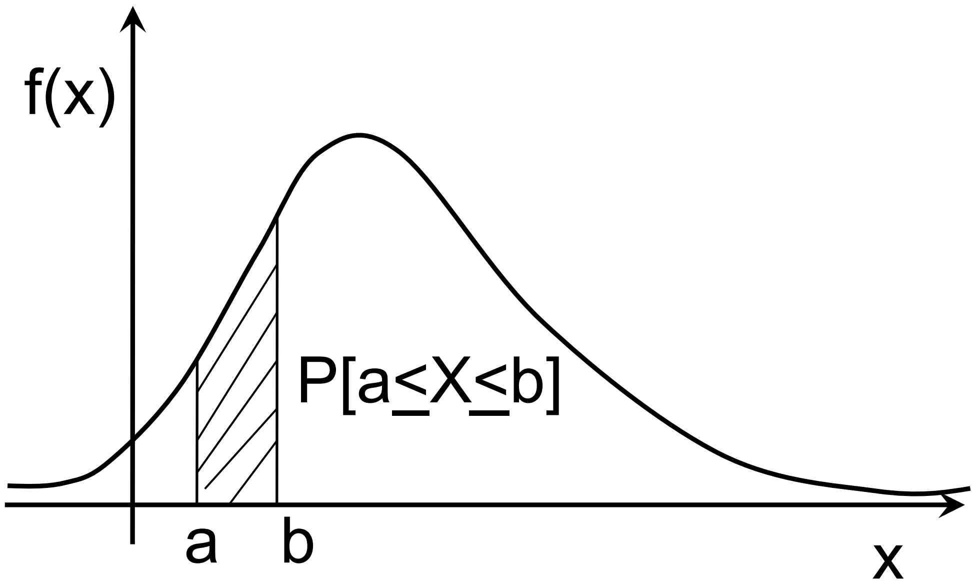
$$P[x_1 \le X \le x_2] = F(x_2) - F(x_1)$$



PDF



CDF



- P[a ≤ X ≤ b] is probability that 'x' takes on a value between 'a' and 'b'
  - equals area under the pdf between 'a' and 'b'

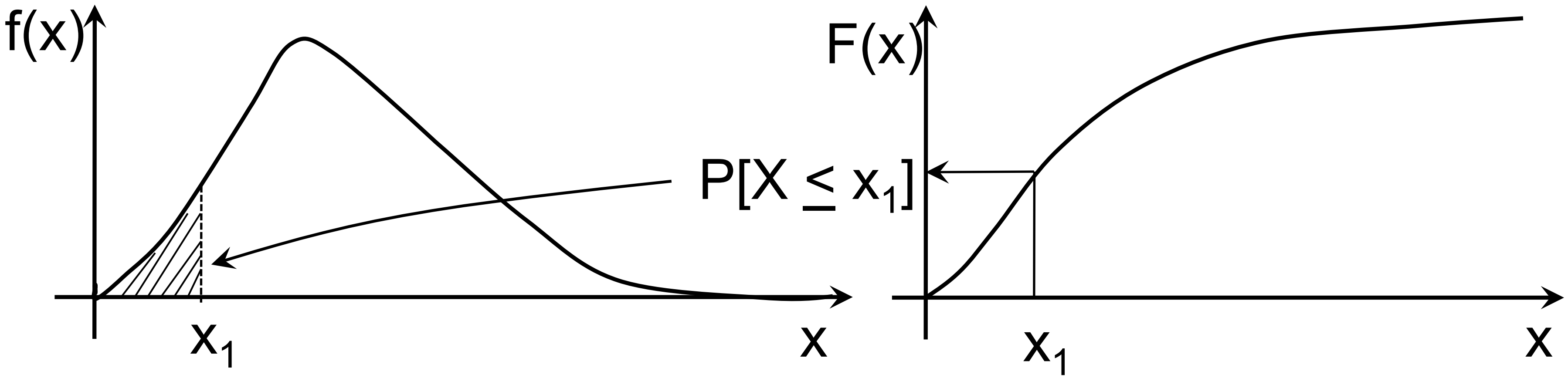
$$= \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx = \int_{a}^{b} f(x)dx$$

- P[a ≤ X ≤ b] = F(b) – F(a)

Continuous RVs

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)dx$$

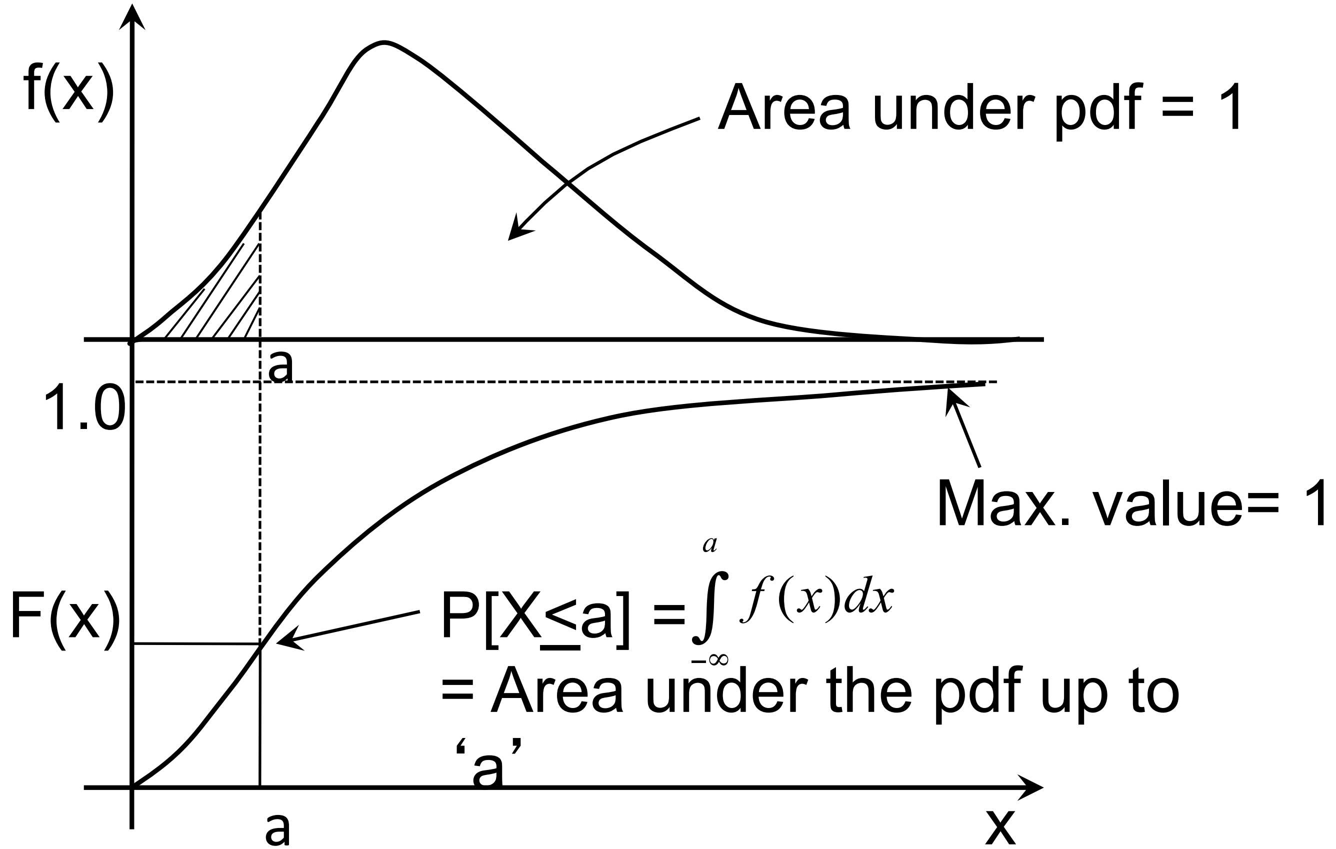
$$f(x) = \frac{dF(x)}{dx}$$



PDF **CDF**

Cumulative Distribution Function

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)\,dx$$



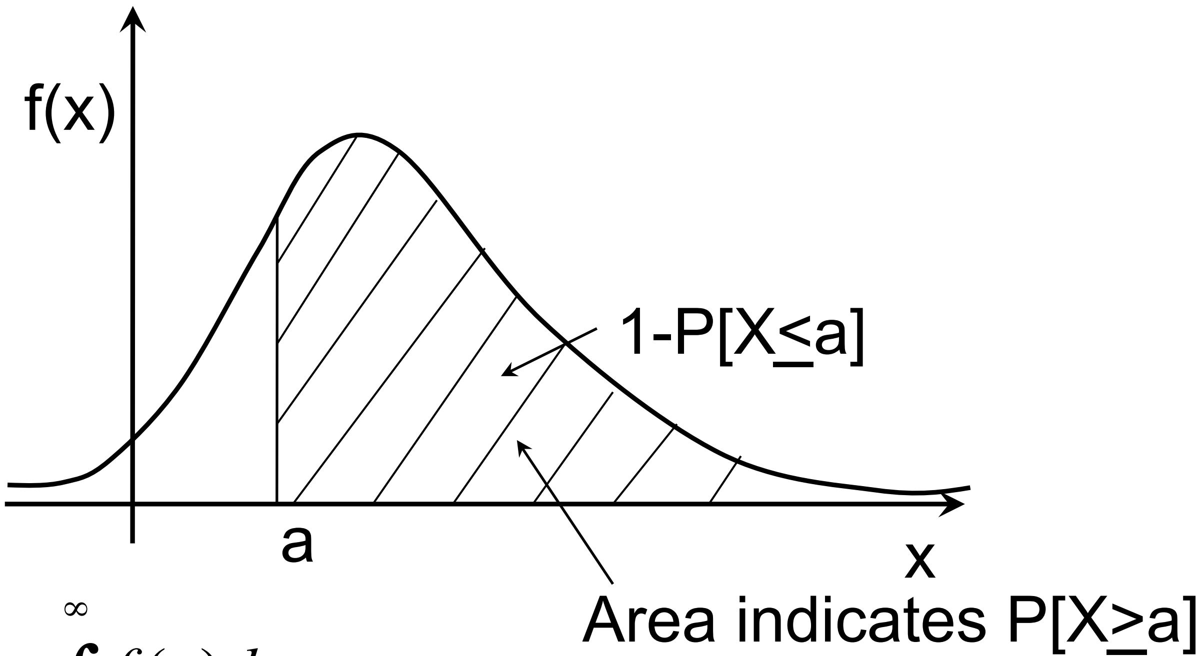
- For continuous RVs, probability of the RV taking a value exactly equal to a specified value is zero

  That is P[X = x] = 0 ; X continuous

$$P[X = d] = P[d \le X \le d] = \int_{d}^{d} f(x)dx = 0$$

- P [x- Δx ≤ X ≤ x+ Δx] ≠ 0
- Because P[X = a] = 0 for continuous r.v.

$$P[a \le X \le b] = P[a < X < b] = P[a \le X < b] = P[a < X \le b]$$



$$P[X \geq a] = \int_a^{\infty} f(x)dx$$

$$= \int_{-\infty}^{\infty} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= 1 - F(a)$$

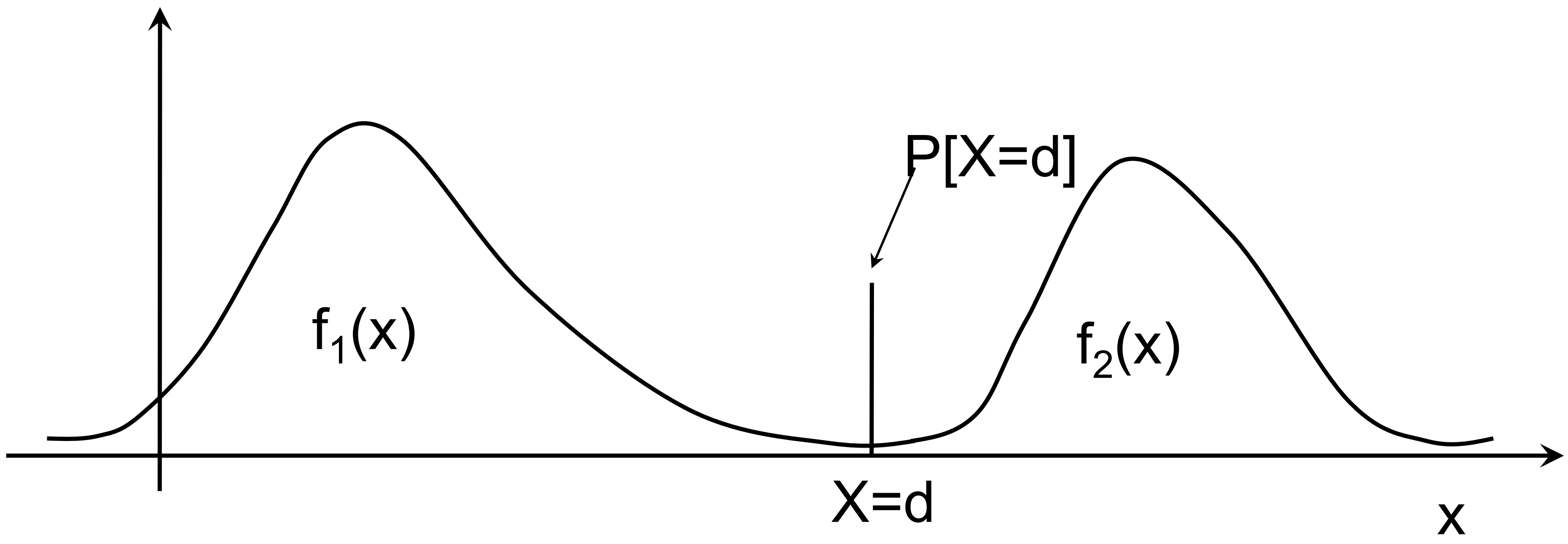
$$= 1 - P[X \leq a]$$

$$P[x \geq a] = 1 - P[x \leq a]$$

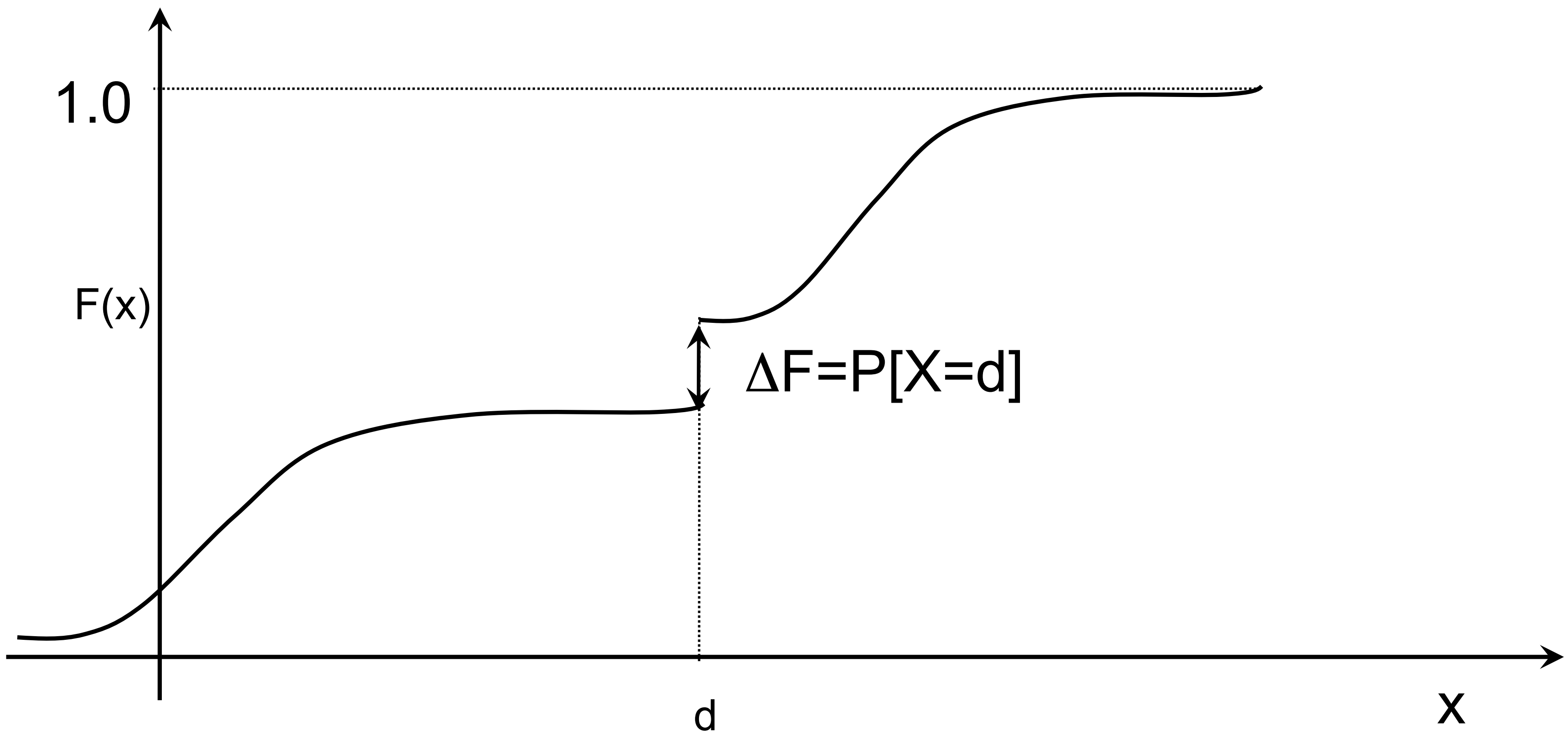
Mixed Distributions

- $P[X = d] \neq 0$
- A finite probability associated with a discrete event X = d
- At other values that 'X' can assume, there may be a continuous distribution.
  - e.g., probability distribution of rainfall during a day: there is a finite probability associated with a day being a non-rainy day, i.e., $P[X=0]$, where x is rainfall during a day; and for $x \neq 0$, the r.v. has a continuous distribution ;

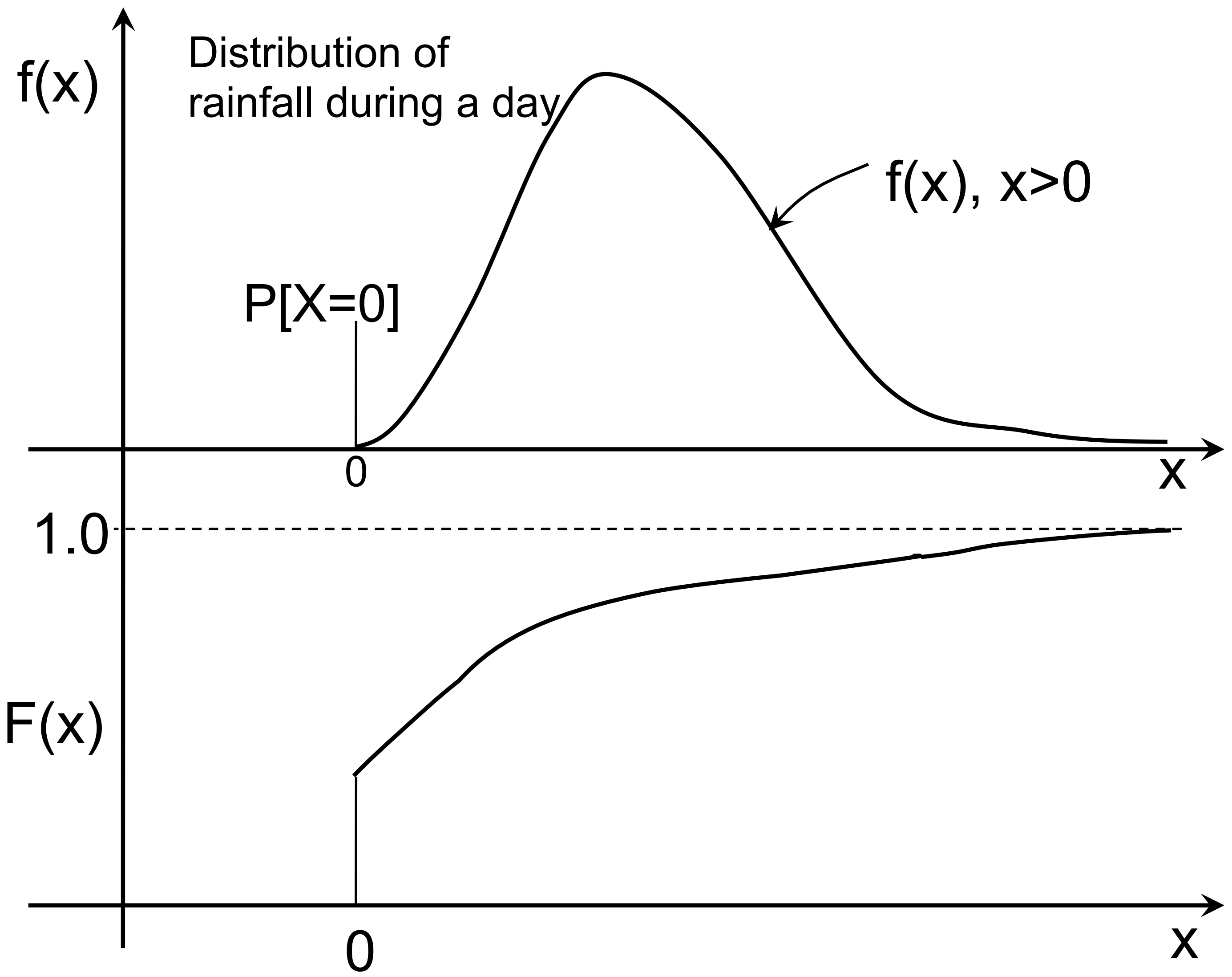
Mixed Distributions



$$\int_{-\infty}^{d} f_1(x)dx + P\left[x = d\right] + \int_{d}^{\infty} f_2(x)dx = 1.0$$



In this case, $P[X < d] \neq P[X \leq d]$



$F(0) = P\ [X \leq 0] = P(X=0)$, in this case.

Example Problem

$f(x) = a.x^2 \quad 0 \leq x \leq 4$

$\quad = 0 \quad$ otherwise

1. Determine the constant 'a'

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{\infty} a.x^2 dx = 1$$

$$a\left[\frac{x^3}{3}\right]_0^4 = 1$$

Gives $a = 3/64$ and $f(x) = 3x^2/64; \quad 0 \leq x \leq 4$

2. Determine F(x)

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{0}^{x} f(x)dx$$

$$= \int_{0}^{x} \frac{3x^2}{64} dx$$

$$= \frac{3}{64}\left[\frac{x^3}{3}\right]_0^x$$

$$F(x) = \frac{x^3}{64} \qquad 0 \le x \le 4$$

Then, for example,

$$P[X \le 3] = F(3) = \frac{3^3}{64} = \frac{27}{64}$$

$P[X \leq 4] = F(4) = 1.0$

$P[1 \leq X \leq 3] = F(3) - F(1) = 26/64$

P [X>6] – From the definition of the pdf, this must be zero

$P[X>6] = 1 - P[X \leq 6]$

$$= 1 - \left[\int_{-\infty}^{0} f(x)dx + \int_{0}^{4} f(x)dx + \int_{4}^{6} f(x)dx \right]$$

$= 1 - [\quad 0 \quad + \quad 1.0 \quad + \quad 0 \quad]$

$= 0$

Example problem

Consider the following pdf

$$f(x) = \frac{1}{5}e^{-x/5} \qquad x>0$$

1. Derive the cdf
2. What is the probability that x lies between 3 and 5
3. Determine ‘x’ such that $P[X \leq x] = 0.5$
4. Determine ‘x’ such that $P[X \geq x] = 0.75$

1. CDF:

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{0}^{x} f(x)dx$$

$$= \int_{0}^{x} \frac{1}{5} e^{-x/5} dx$$

$$= \left[-e^{-x/5} \right]_{0}^{x}$$

$$F(x) = \left[1 - e^{-x/5} \right]$$

2. P[3 $\leq$ X $\leq$ 5] = F(5) - F(3)

= 0.63 – 0.45

= 0.18

3. Determine ‘x’ such that P[X ≤ x] = 0.5

$$\left[1-e^{-x/5}\right]=0.5$$

$$-x/5=\ln 0.5$$

$$x=3.5$$

4. Determine ‘x’ such that P[X ≥ x] = 0.75

$$\mathrm{P[X \geq x]}=1-P\left[X \leq x\right]=0.75$$

$$1-\left[1-e^{-x/5}\right]=0.75$$

$$e^{-x/5}=0.75$$

$$-x/5=\ln 0.75$$

$$x=1.44$$