$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = P[X \le x]$$

$$P[a \leq x \leq b] = \int_a^b f(x)\,dx$$

$$= F(b) - F(a)$$

$$\mathrm{Cov}(X,Y) = \Gamma_{XY}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y)\, f(x,y)\, dx\, dy$$

$X, Y \rightarrow$ independent

$$f(x,y) = g(x) \cdot h(y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) \cdot g(x) \cdot h(y)$$

$g(x) \rightarrow$ m.d of X

$h(y) \rightarrow$ m.d of Y

$$\sigma_{xy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)\, g(x)\, dx\, (y-\mu_y)\, h(y)\, dy$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)\, g(x)\, dx \int_{-\infty}^{\infty} (y-\mu_y)\, h(y)\, dy$$

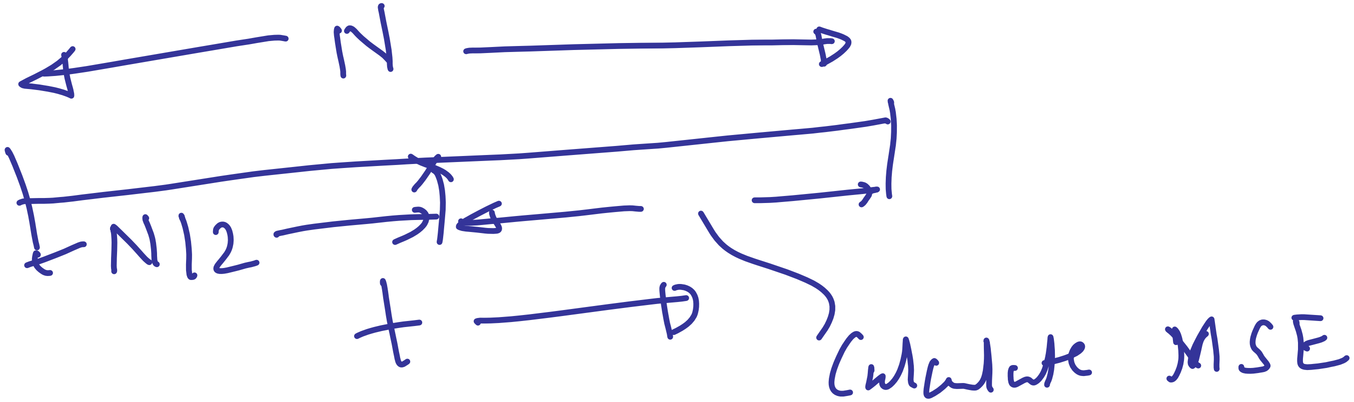
$$\sigma_{xy} = 0 \cdot 0 = 0$$ if X & Y are indepnd.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

AR(2)

$$Z_t = \frac{X_t - \bar{X}_t}{S_t}$$

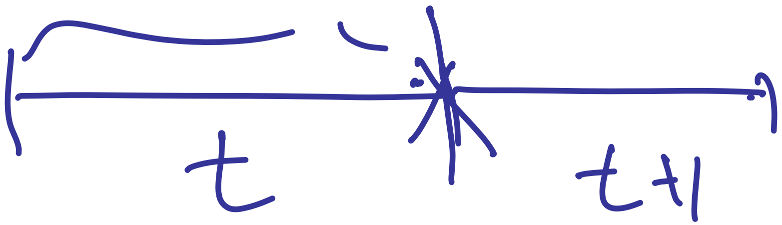
$$X_t = \theta_1 e_{t-1} + e_t$$



AR(1)

$$X_t = \phi_1 X_{t-1} + \epsilon_t$$

Noise



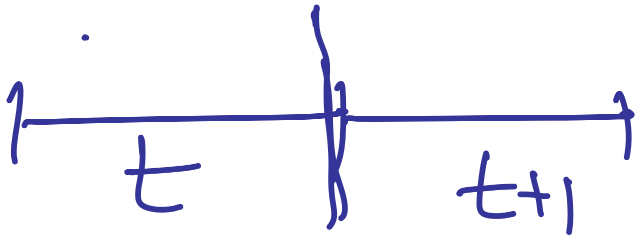
$$x_{t+1} = \phi_1 x_t + \epsilon_{t+1}$$

$$\hat{X}_{t+1} = E[X_{t+1}] = \phi_1 E[X_t] + \underline{\underline{E[\epsilon_{t+1}]}}$$

$$\hat{X}_{t+1} = \phi_1 E[X_t] + 0$$

$$= \phi_1 E[X_t]$$

ARMA(1,1)



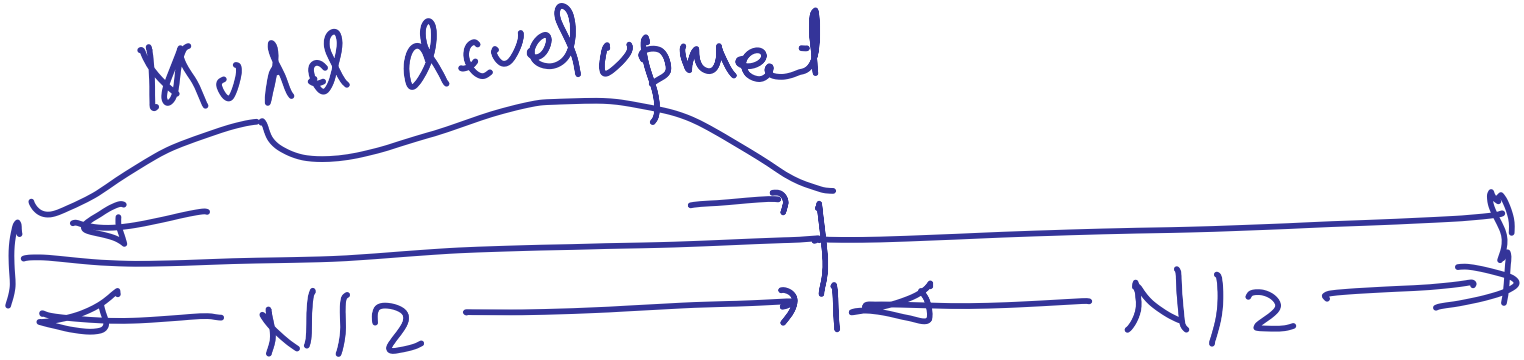
$$X_{t+1} = \phi_1 X_t + \theta_1 \underline{\underline{e_t}} + e_{t+1}$$

Actual value of residual

$$E[X_{t+1}] = \phi_1 E[X_t] + \theta_1 E[e_t] + e_{t+1}$$

$$\hat{X}_{t+1} = \phi_1 E[X_t] + \theta_1 e_t + E[e_{t+1}]$$

$E[e_{t+1}] \to 0$



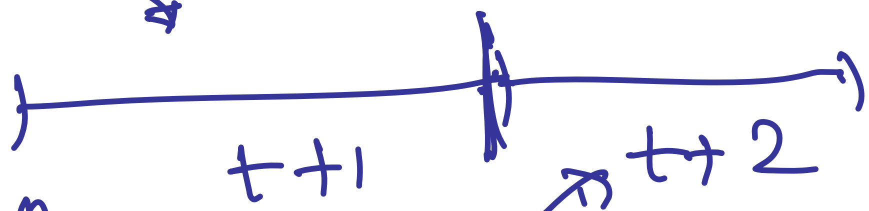
$50 \times 12 = 600$

No. of years — No. of months

$X_{t+1} \longrightarrow$ Known data Value

$\hat{X}_{t+1} \longrightarrow$ Forecasted Value

$$e_{t+1} = X_{t+1} - \hat{X}_{t+1}$$

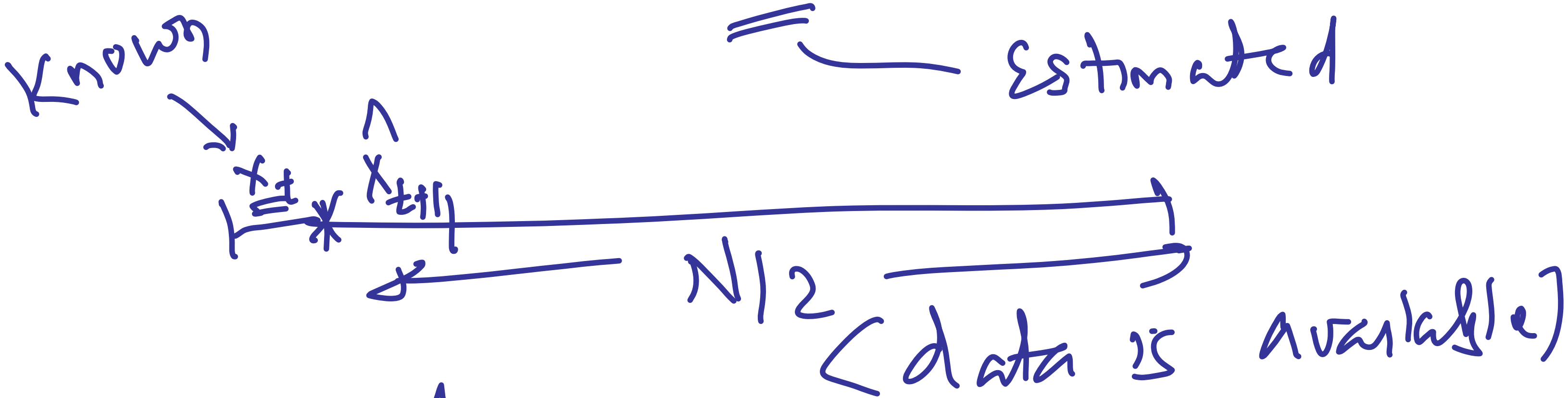


$$\hat{X}_{t+2} = \phi_1 \underbrace{X_{t+1}}_{\text{known}}$$

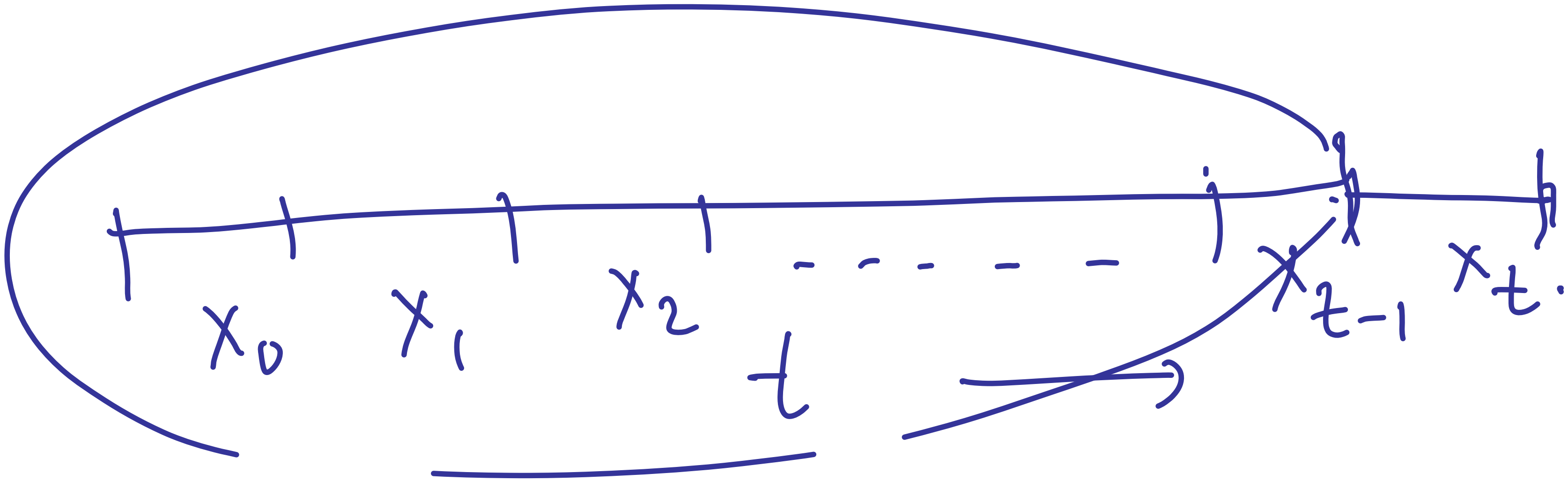
$$\hat{x}_{t+1} = \phi_1 \bar{x}_t$$

$$= \phi_1 E[x_t]$$

ϕ_1 — estimated

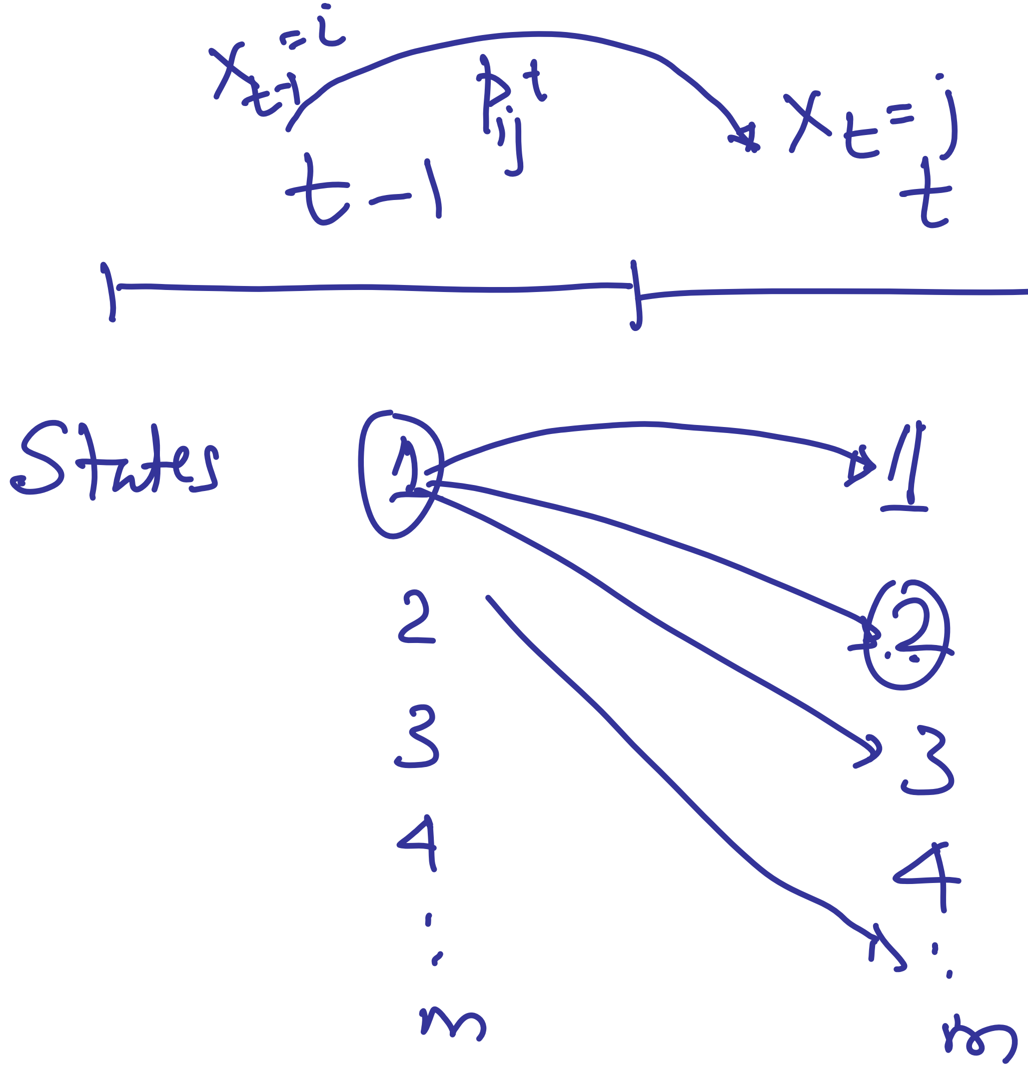


$$\hat{x}_{t+1} = \phi_1 x_t$$



$$P[x_t / x_{t-1} \ldots x_0] = P[x_t / x_{t-1}]$$

One Step Markov Chain



$$P[X_t = j \mid X_{t-1} = i] = p_{ij}^t$$

→ transition probability

$$\sum_{j=1}^{m} p_{ij}^t = 1 \quad \forall i$$

TPM

Transition Probability Matrix

$$
P = \begin{array}{c|cccccc|}
t-1 \downarrow \;\backslash\; t \rightarrow & 1 & 2 & 3 & 4 & \cdots & m \\
\hline
1 & p_{11} & p_{12} & p_{13} & & & p_{1m} \\
2 & p_{21} & p_{22} & \cdots & & \cdots & p_{2m} \\
3 & \vdots & & & & & \\
\vdots & \vdots & & & & & \\
m & p_{m1} & p_{m2} & & & \cdots & p_{mm}
\end{array}_{m \times m}
$$

Jun $t=1$	Jul $t=2$	p_{11}
3	2	
1	4	
3	6	
4	3	
1	1	
1	1	
0		

30 → class 1

15

State 1 : 0-100

2 : 100-200

3

10

$$p^{(1)} = p^{(0)} \times P$$

$$p^{(2)} = p^{(1)} \times P$$

$$\vdots$$

$$p^{(n+1)} = \underline{\underline{p}}^{(n)} \times P$$

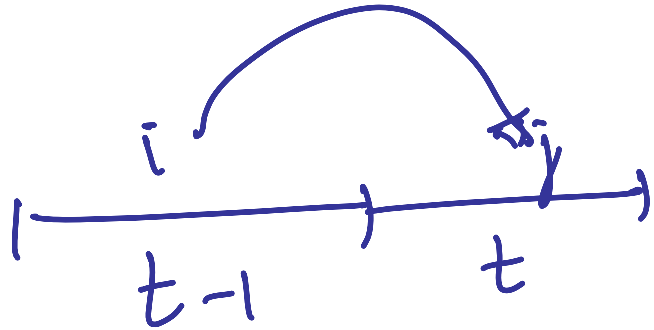
$$\underbrace{p^{(n+1)} = p^{(n+1)}} \times P$$

$$p = p^{(n+1)} = p^{(n)}$$

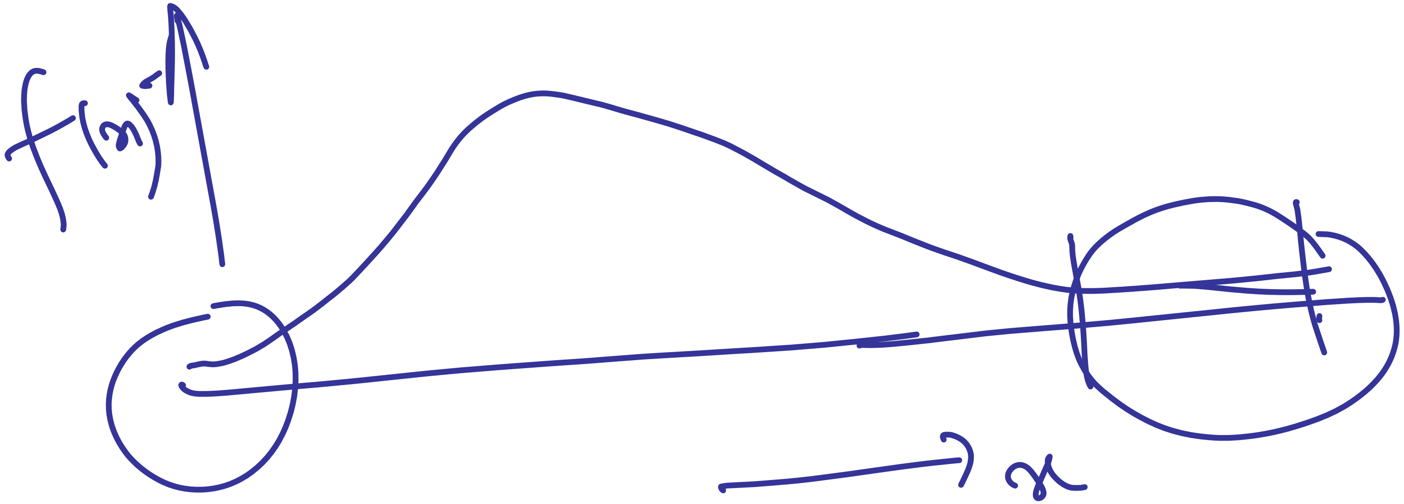
for large n

$$\underline{\underline{p}} = \underbrace{\underline{\underline{p}} \times p}_{\text{Steady State}}$$

t-1 ↓ \ t →	1	2
1	0.7	0.3
2	0.4	0.6



$$
P = \begin{array}{c|cccc} t-1 \backslash t & 1 & 2 & \cdots & m \\ \hline 1 & p_{11} & p_{12} & \cdots & p_{1m} \\ 2 & p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & & \\ m & p_{m1} & p_{m2} & \cdots & p_{mm} \end{array}_{m \times m}
$$

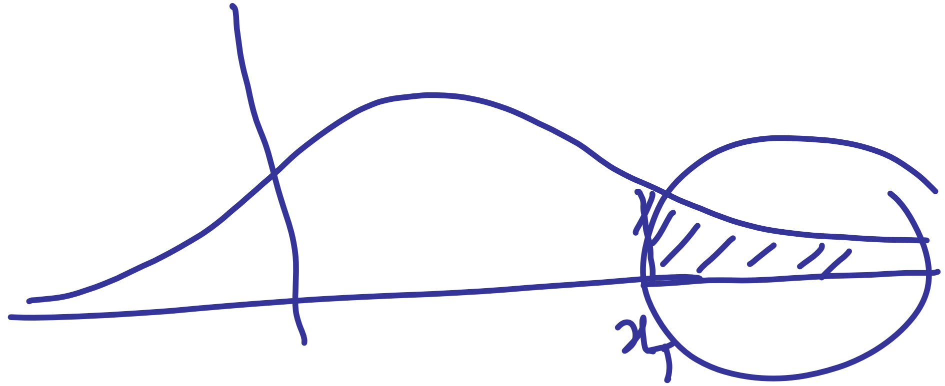


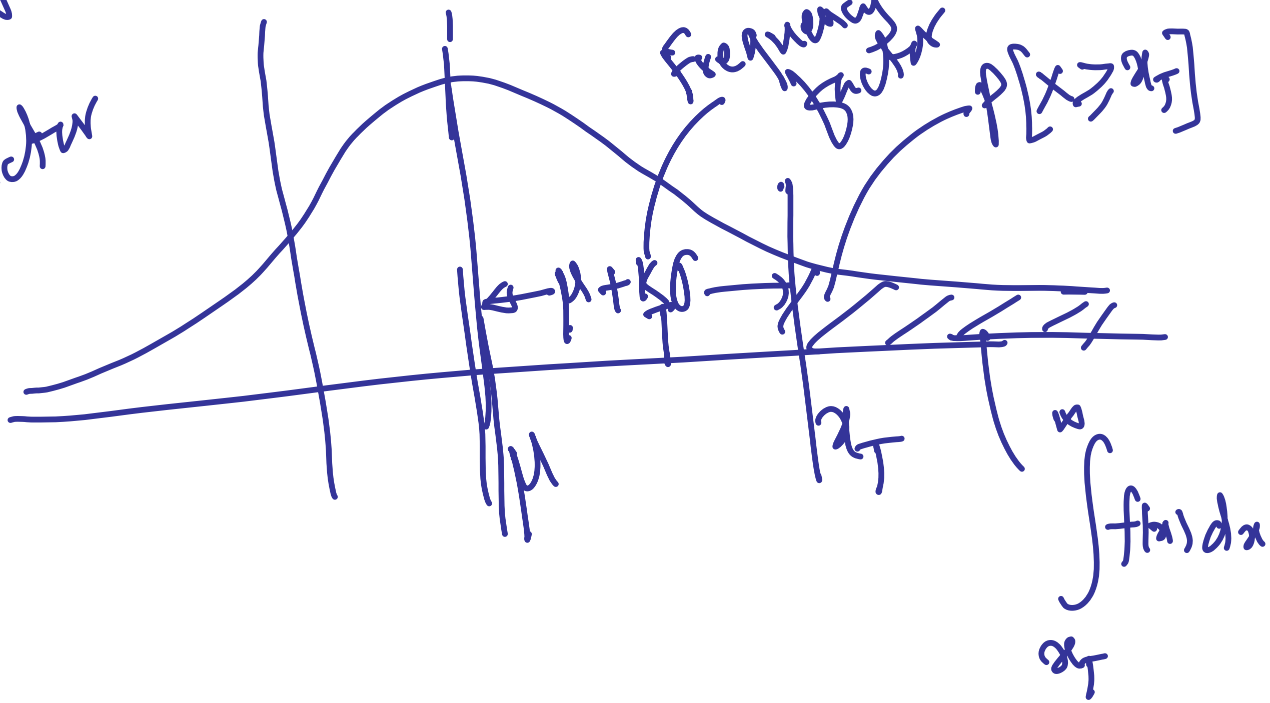
X : Streamflow

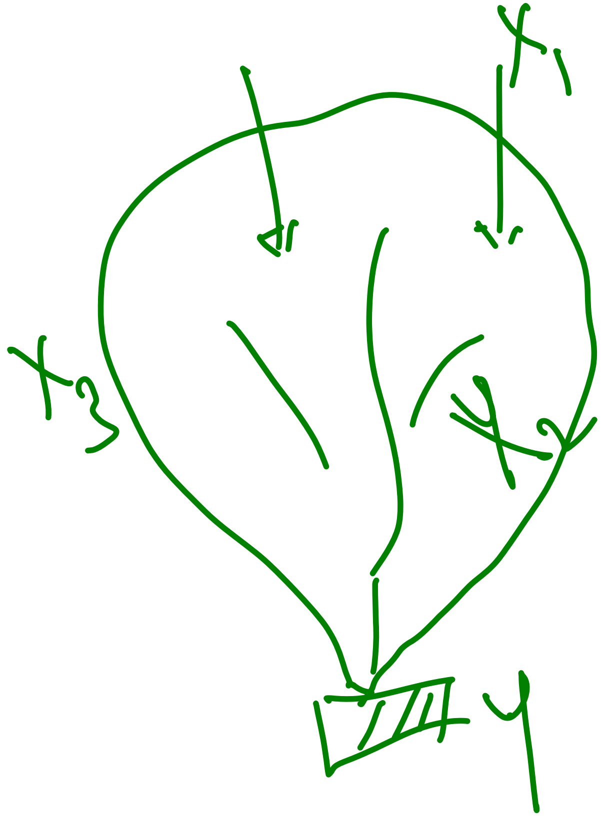
$P[X > x_1]$

$F(x)$ → Not readily invertible

K_T : Frequency factor







$$\underline{\underline{y}} = ax + b$$

$$\underline{\underline{y}} = \beta_1 x_1 + \beta_2 x_2 + \cdots \beta_p x_p$$

$$[\quad]_{n \times p} \, [\quad]_{p \times q} = [\quad]_{n \times q}$$

Square Matrix $\longrightarrow$ Eigenvectors

Not all square matrices have eigenvectors

$n \times n$ Square matrix

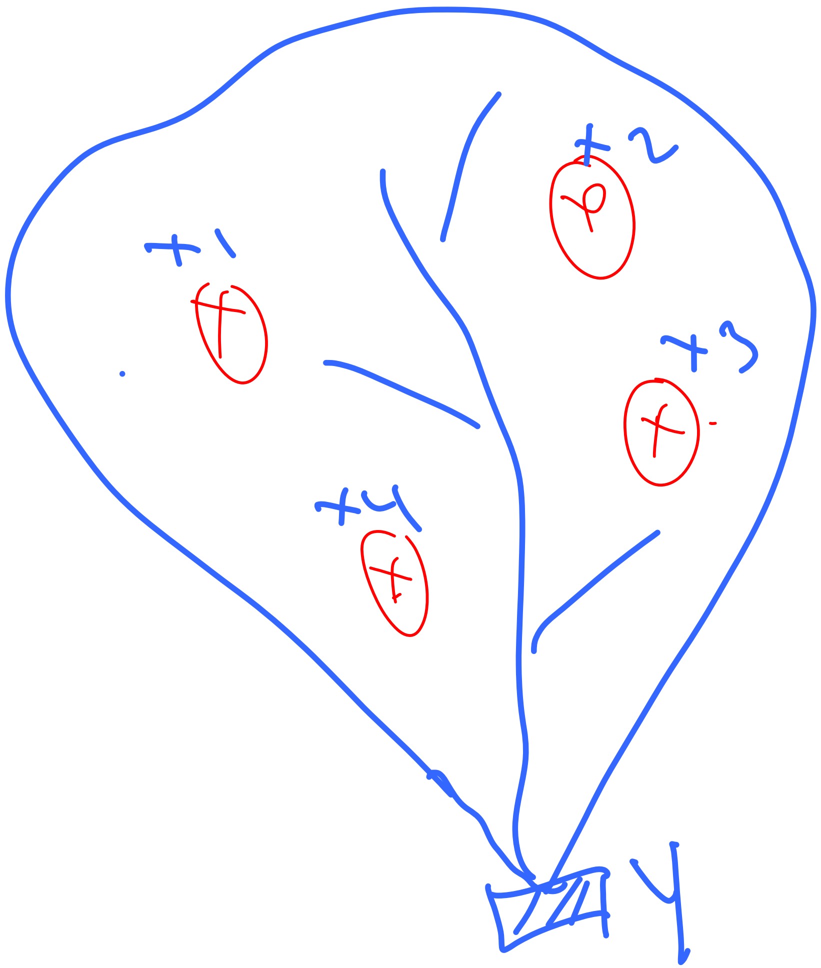
→ n eigenvectors

→ n eigen values.

variables → X_1 X_2 X_3

no. of observations ↓ 1, 2, 3, ⋮, n

$n \times p$



$$y = f(x_1, x_2, \ldots, x_p)$$

$$X_t' = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{Bmatrix}_t$$

$$\Sigma = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ p \end{array} \begin{array}{cccc} 1 & 2 & \cdots & p \\ \hline \rho_1(1) & 0 & \cdots & 0 \\ 0 & \rho_2(1) & & \\ 0 & 0 & \ddots & \\ & & & \ddots \\ 0 & \cdots & & \rho_p(1) \end{array}$$

$p \times p$

$$
\Gamma = \begin{bmatrix} \sqrt{1-\rho_1^2} & 0 & 0 & 0 & 0 \\ \vdots & \sqrt{1-\rho_2^2} & & & \\ \vdots & & \ddots & & \\ 0 & \cdots & \cdots & 0 & \sqrt{1-\rho_p^2} \end{bmatrix}_{p \times p}
$$

$$
\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_p \end{bmatrix}_{p \times 1}
$$

$$D_\lambda^{1/2} = \begin{bmatrix} \lambda_1^{1/2} & 0 & 0 & 0 \\ 0 & \lambda_2^{1/2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_p^{1/2} \end{bmatrix}_{p \times p}$$

λ_j : j^{th} largest eigen value of the $p \times p$ correlation matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ p \end{array} \begin{array}{c} 1 \quad 2 \quad \cdots \quad p \\ \begin{bmatrix} \rho_{11}^{*}(0) & & \rho_{1p}^{*}(0) \\ \rho_{21}^{*}(0) & & \vdots \\ \vdots & & \rho_{pp}^{*}(0) \end{bmatrix}_{p \times p} \end{array}$$

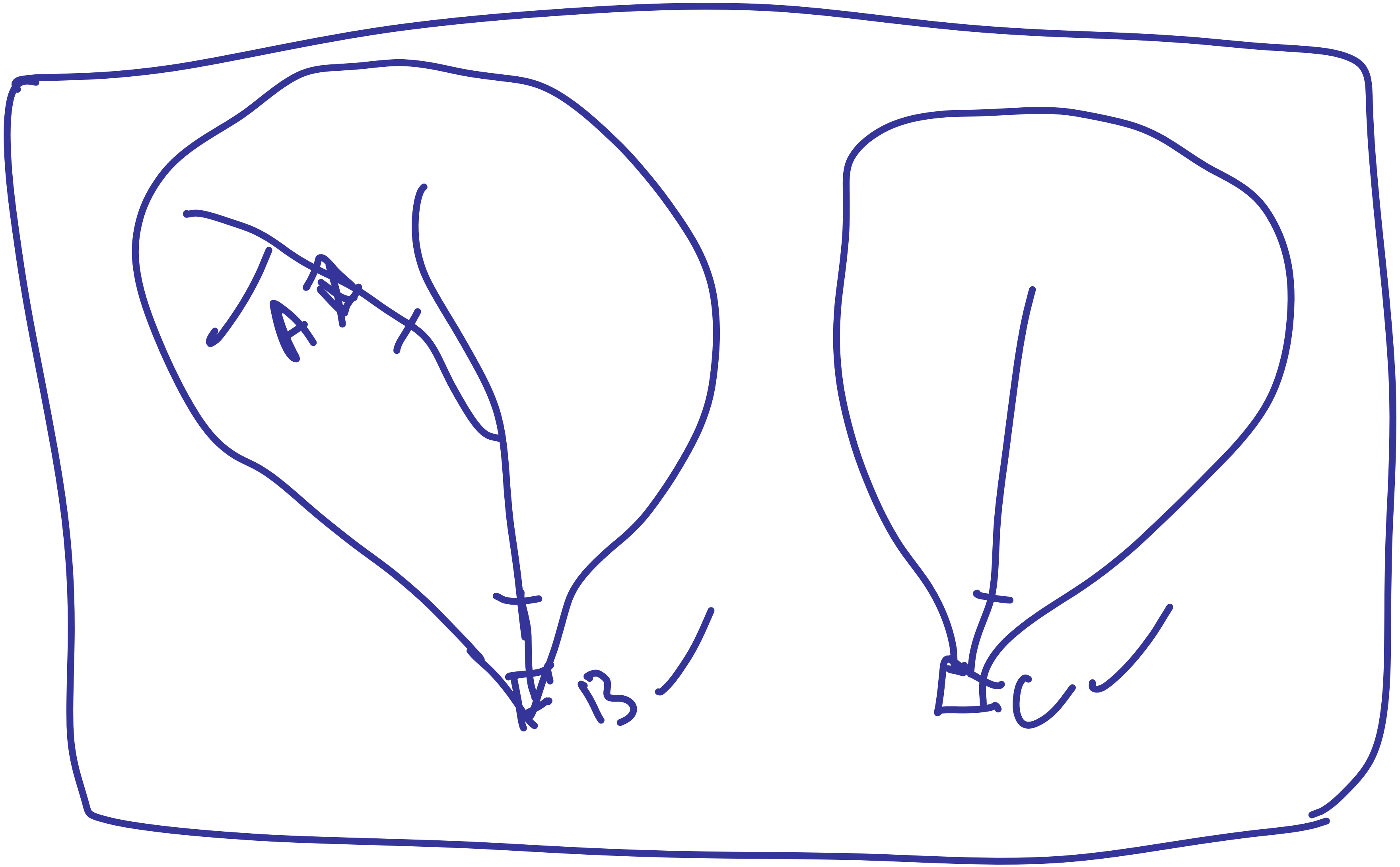
$$A = \begin{bmatrix} \overset{1}{(\;)} & \overset{2}{(\;)} & \cdots & \overset{p}{} \end{bmatrix}_{p \times p}$$

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}_{p \times 1}$$

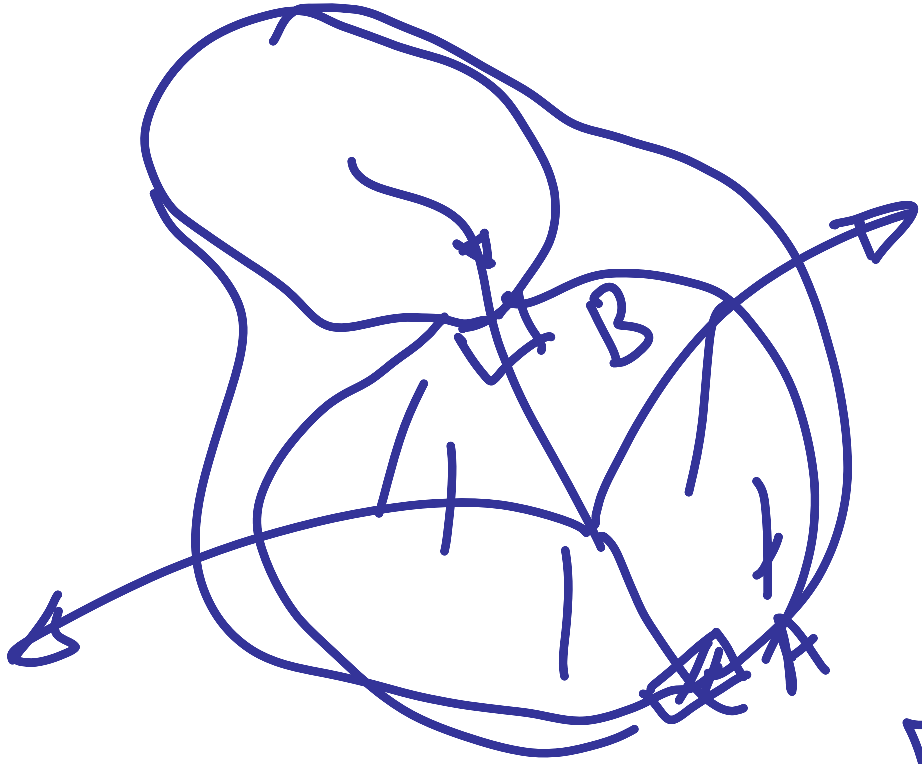
$$e = N(0,1)$$

$$\begin{bmatrix} x(1,t) \\ x(2,t) \\ \vdots \\ x(p,t) \end{bmatrix}_{p\times 1} \begin{bmatrix} x(1,t) & x(2,t) & \cdots & x(p,t) \end{bmatrix}_{1\times p}$$

$$[X_t X_t'] = \begin{bmatrix} x(1,t)x(1,t) & x(1,t)x(2,t) & \cdots & x(1,t)x(p,t) \\ x(i,t)\,x(1,t) & x(1,t)x(2,t) & \cdots & x(1,t)\,x(p,t) \\ \vdots & & & \\ x(p,t)\,x(1,t) & x(p,t)\,x(2,t) & \cdots & x(p,t)\,x(p,t) \end{bmatrix}_{p\times p}$$

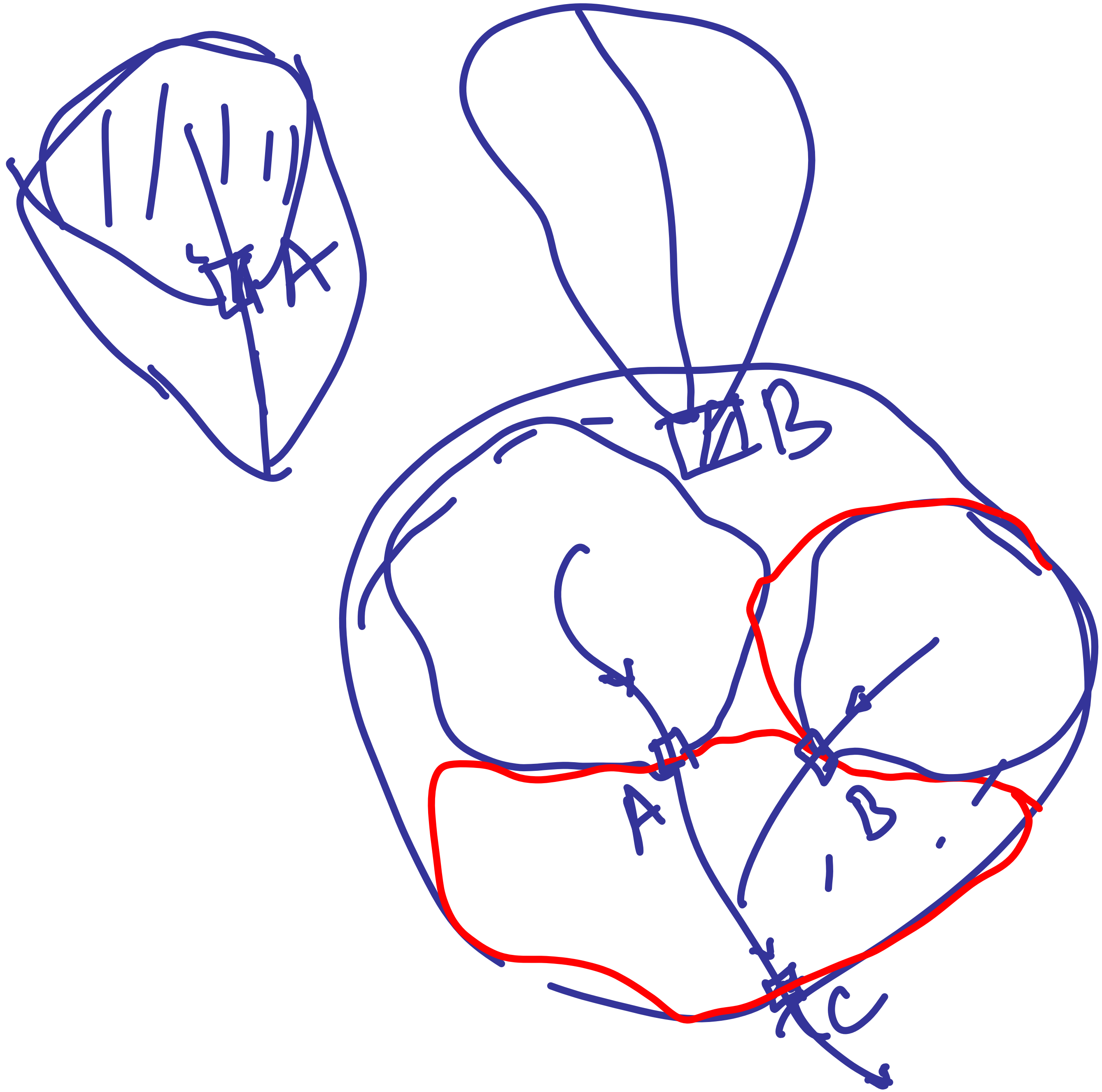


A
B
C



$y_A \neq y_B$

Specific Flows



A
B
A
B
C

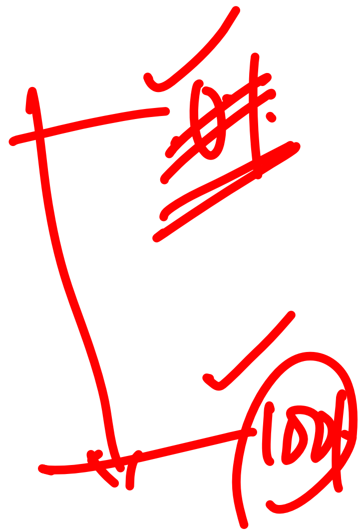
$\{y_t\}$

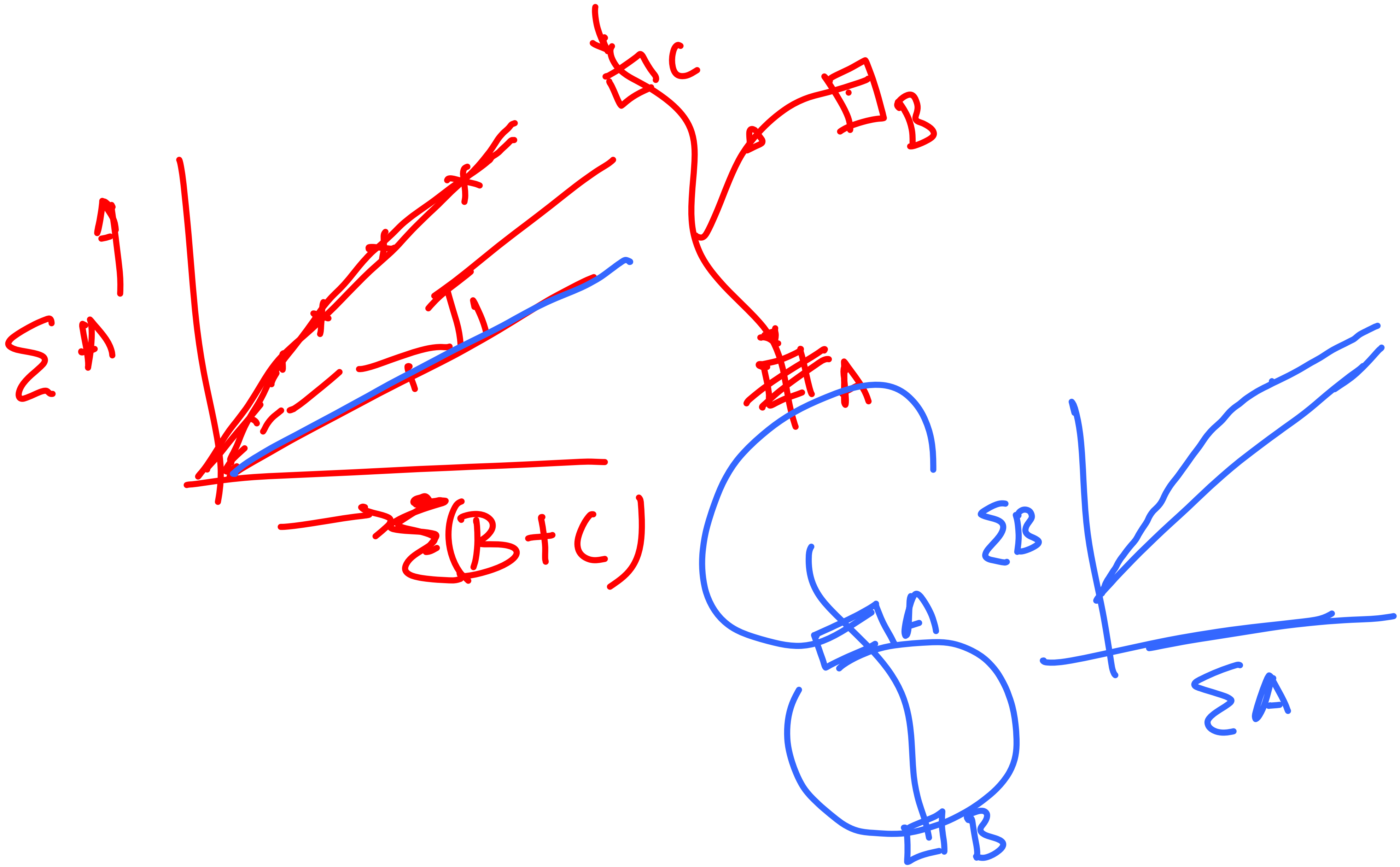
$P[X \geq x] = 0.5$

500

1
2
⋮
600

$\frac{m}{n+1}$

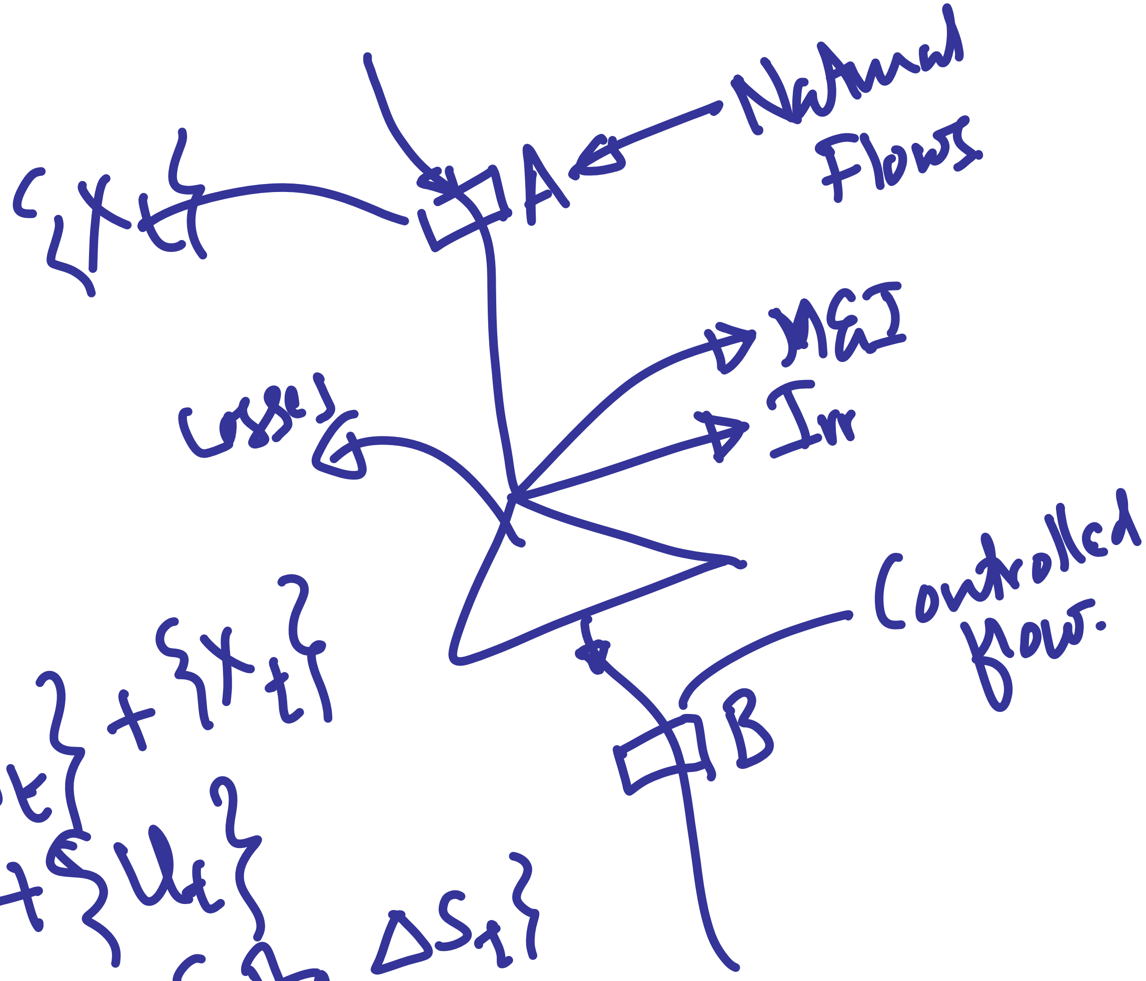


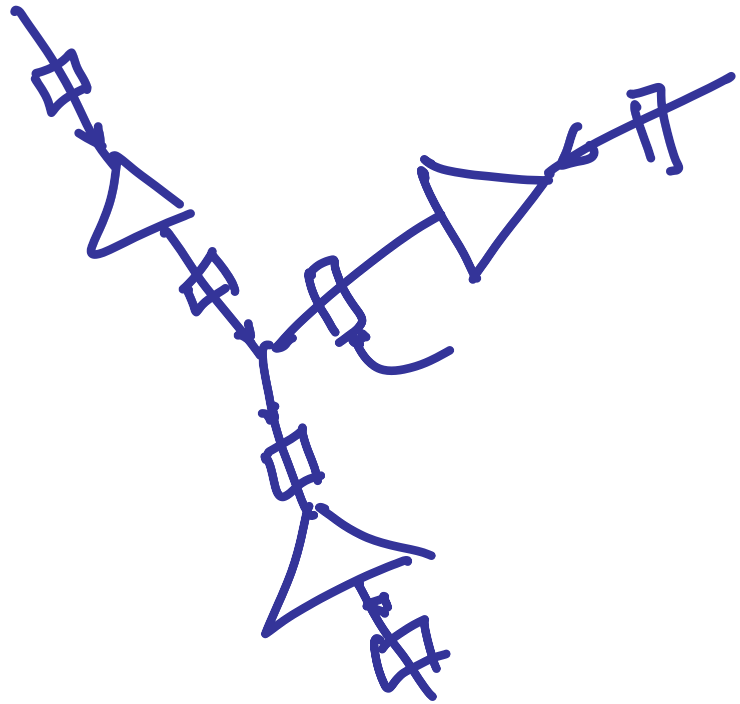


ΣA
Σ(B+C)
C
B
A
ΣB
A
B
ΣA

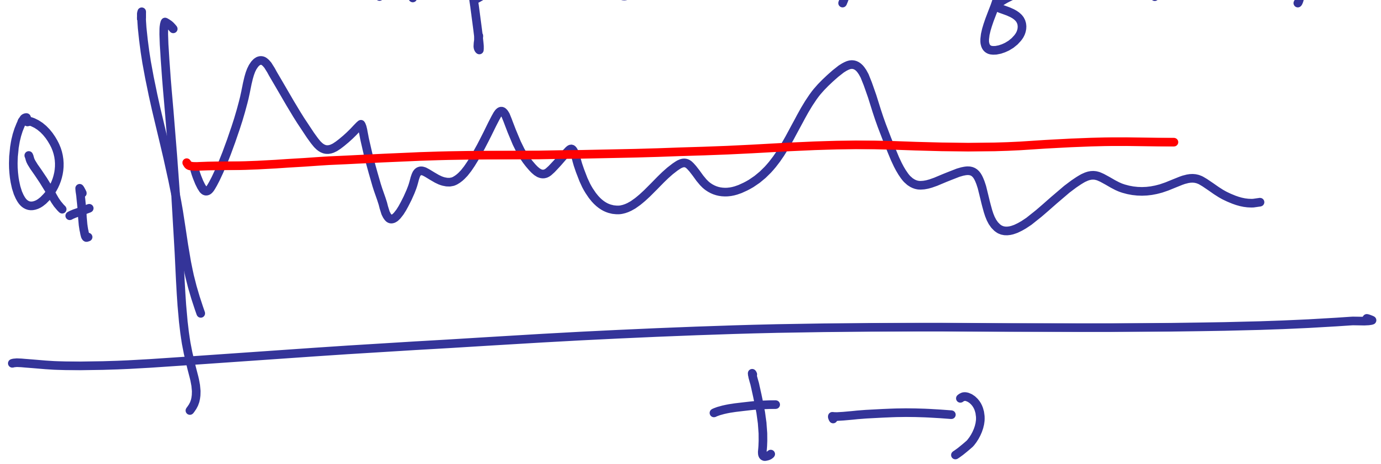
$\{Y_t\}_B$

$$\{Z_t\} = \{Y_t\} + \{X_t\} + \{U_t\} \pm \{\Delta S_t\}$$

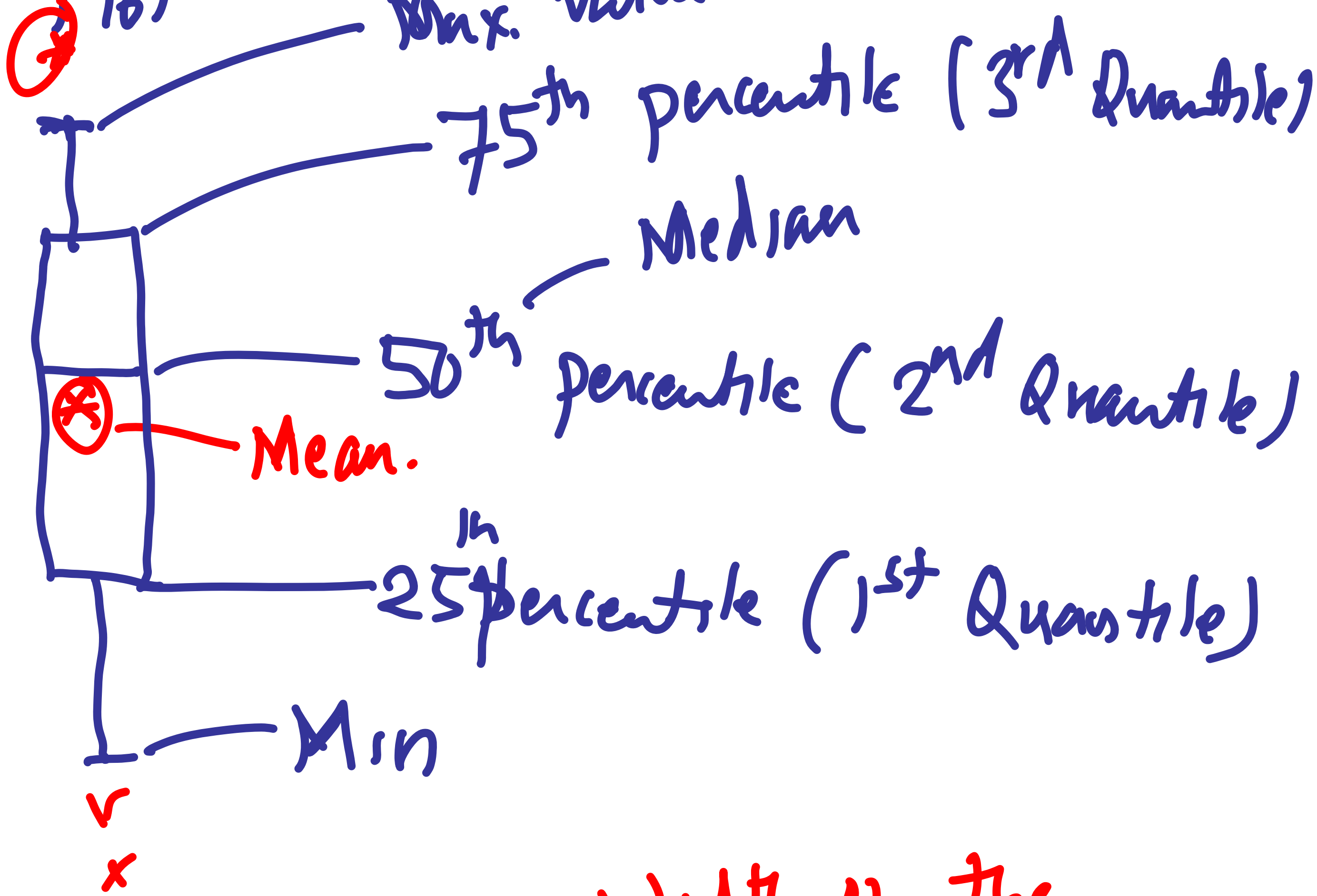




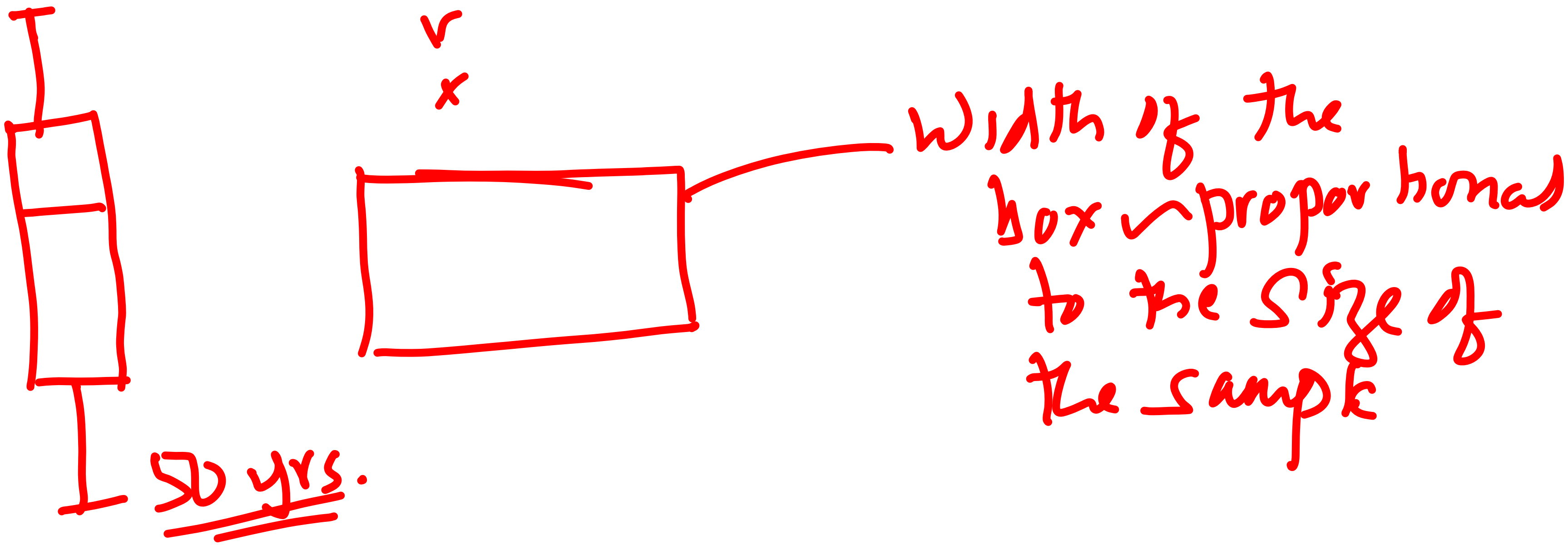
Representation of Uncertainty bands



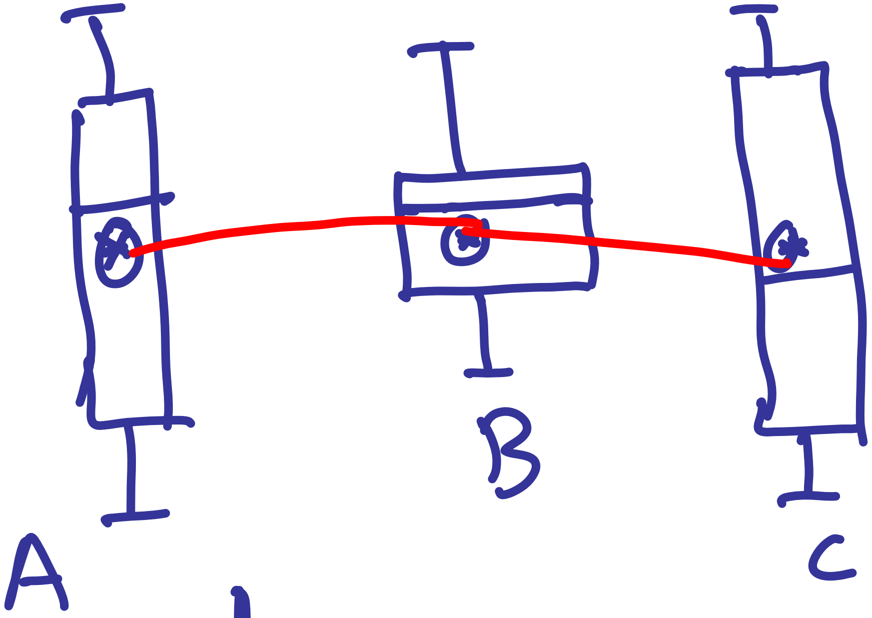
Box - Plot

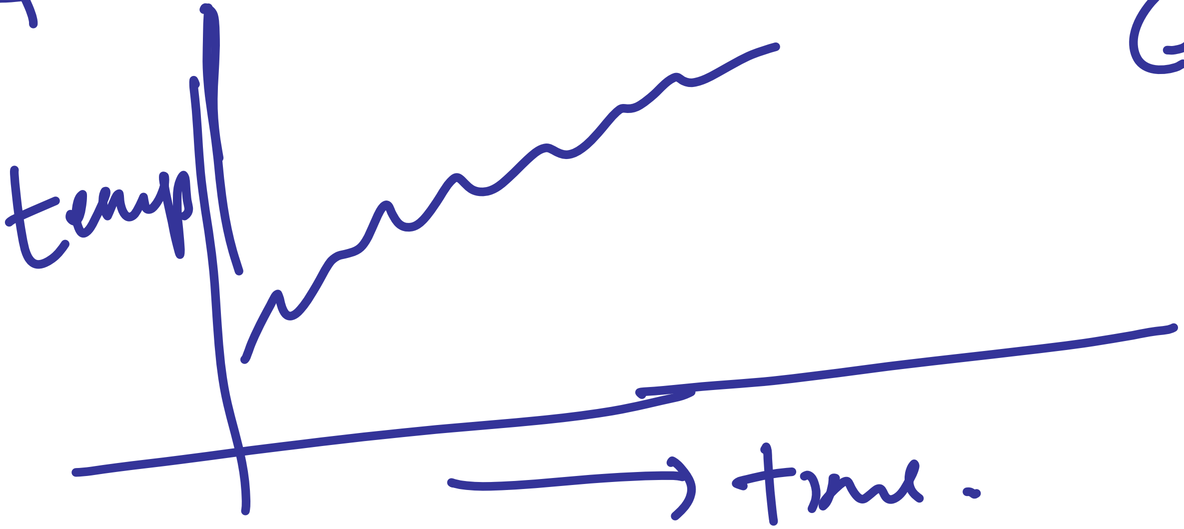


Max. value.
75th percentile (3rd Quartile)
Median
50th percentile (2nd Quartile)
Mean.
25th percentile (1st Quartile)
Min



50 yrs.
Width of the box proportional to the size of the sample





Green House Gas Effect.