Tutorial 4 - Axial Elements (Matrix Methods) CE3310

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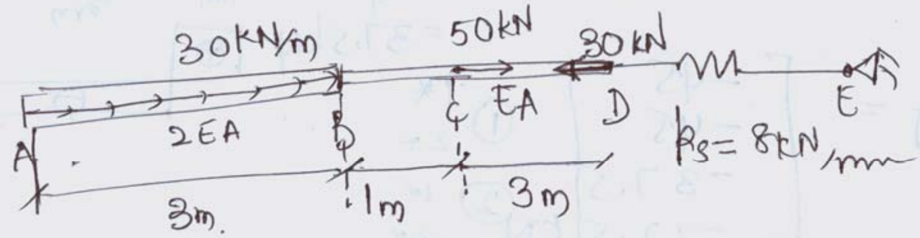
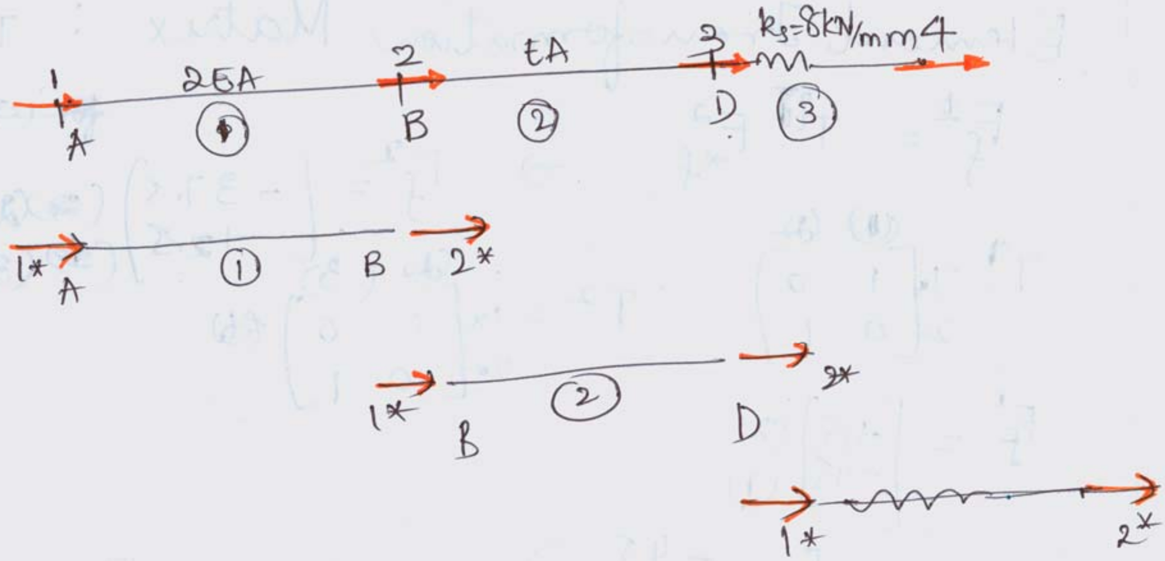
$EA = 100 \times 30$ KN
$= 3,000$ KN.

$\frac{20}{20}$

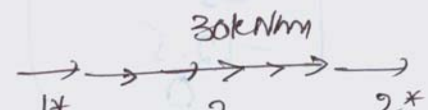
1. (i)



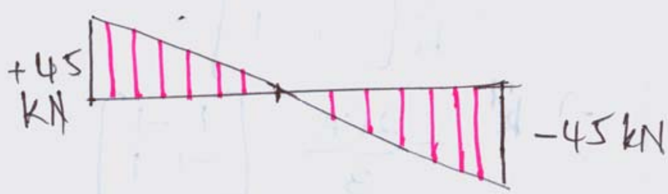


Fixed End Forces:

$$F^1_{*f} = \begin{Bmatrix} -45 \\ -45 \end{Bmatrix} \text{KN}$$



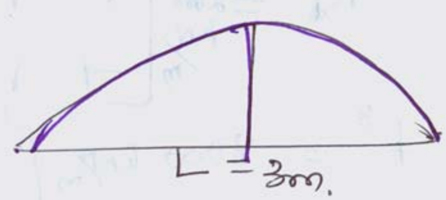
Member Axial force Diagram:



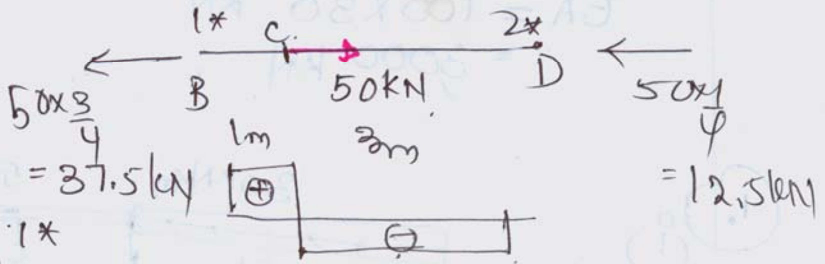
$$\frac{du^*}{dx} = \frac{N(x^*)}{EA}$$

$$u^* = \frac{1}{EA}\int_0^{x^*} (45 - 30x^*)\, dx \quad = 45x^* - 15(x^*)^2$$

$$u^*_{max} = \frac{1}{EA}\left(45 \times \frac{1}{2} - \frac{15}{4}\right) = \frac{18.75}{EA}$$



$$F^2_{*f} = \begin{bmatrix} -37.5 \\ -12.5 \end{bmatrix} kN$$



$$F_{*f} = \begin{bmatrix} -45 \\ -45 \\ -37.5 \\ -12.5 \end{bmatrix} kN \quad \begin{array}{l} ①\ 1* \\ \quad 2* \\ ②\ 1* \\ \quad 2* \end{array}$$

Element Transformation Matrix : $T^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ~~for i=1,2~~

$$F_f^2 = T^{iT} F^2_{*f} \Rightarrow F_f^2 = \begin{bmatrix} -37.5 \\ -12.5 \end{bmatrix} \begin{array}{l} (2) \\ (3) \end{array}$$

$$T^1 = \begin{array}{c|cc} & (1) & (2) \\ 1* & 1 & 0 \\ 2* & 0 & 1 \end{array} \qquad T^2 = \begin{array}{c|cc} & (2) & (3) \\ 1* & 1 & 0 \\ 2* & 0 & 1 \end{array}$$

$$F_f^1 = \begin{bmatrix} -45 \\ -45 \end{bmatrix} \begin{array}{l} (2) \\ (1) \end{array}$$

$$\Rightarrow F_f = \begin{bmatrix} -45 \\ -45-37.5 \\ -12.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -45 \\ -82.5 \\ -12.5 \\ 0 \end{bmatrix} kN$$

Element Stiffness Matrix

$$k^i_* = \frac{EA}{L_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad k^2_* = \frac{EA}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow k^1 = \frac{2EA}{3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad k^3_* = 8\ kN/mm \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$T^{iT} k_{i*} T^i$:

$$k^1 = 2000\ kN/m \begin{array}{c} ① \quad ② \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} ① \\ ② \end{array} \qquad k^2 = 750 \begin{array}{c} ② \quad ③ \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} ② \\ ③ \end{array}$$

$$k^3 = 8000\ kN/m \begin{array}{c} ③ \quad ④ \\ \begin{bmatrix} 1 & -1 \end{bmatrix} \end{array} ③$$

$$
k = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2000 & -2000 & 0 & 0 \\ -2000 & 2000+750 & -750 & 0 \\ 0 & -750 & (750+8000) & -8000 \\ 0 & 0 & -8000 & 8000 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \ kN/m
$$

$$
= \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2000 & -2000 & 0 & 0 \\ -2000 & 2750 & -750 & 0 \\ 0 & -750 & 8750 & -8000 \\ 0 & 0 & -8000 & 8000 \end{bmatrix} \end{array} \ kN/m
$$

Active coordinates ①, ②, ③, restrained ④

Global force vector: $\{F\} = \begin{Bmatrix} 0 \\ 0 \\ -30 \\ F_4 = ? \end{Bmatrix}$; $D = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 = 0 \end{Bmatrix}$
(Load Input)

$$
\{F\} - \{F_f\} = k\{D\}
$$

$$
-\begin{Bmatrix} -45 \\ -82.5 \\ -12.5 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -30 \\ F_4 \end{Bmatrix} = \left[\begin{array}{ccc|c} 2000 & -2000 & 0 & 0 \\ -2000 & 2750 & -750 & 0 \\ 0 & -750 & 8750 & -8000 \\ \hline 0 & 0 & -8000 & 8000 \end{array}\right] \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 = 0 \end{Bmatrix}
$$

$$
\Rightarrow F_A = k_{AA} D_A + (k_{AR}) \cancel{D_R} \xrightarrow{} 0 = k_{AA} D_A .
$$

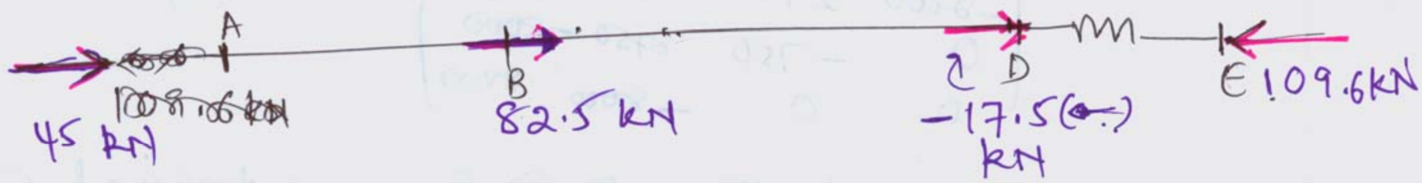
$$
\Rightarrow \begin{Bmatrix} +45 \\ 82.5 \\ -17.5 \end{Bmatrix} = k_{AA} \{D_A\}
$$

$$
\{D_A\} = k_{AA}^{-1} \{F_A\} = \begin{Bmatrix} 0.20625 \end{Bmatrix}
$$

$F_4 = -8000\,D_3 \quad kN/m \times m$

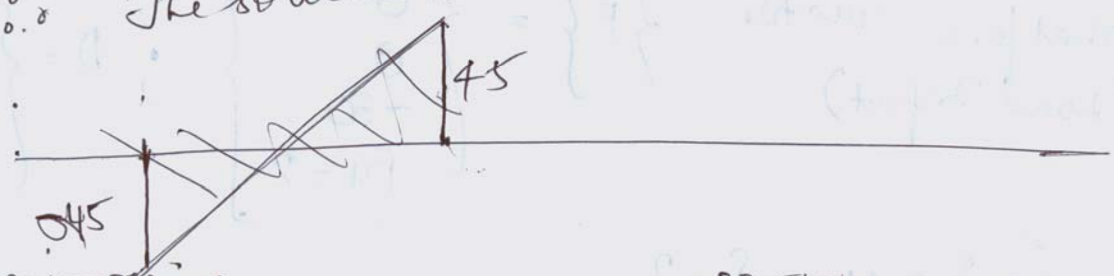
$= -8000 \times 0.0137 = -109.6\ kN$

$$\boxed{F_4 = -109.6\,kN}$$

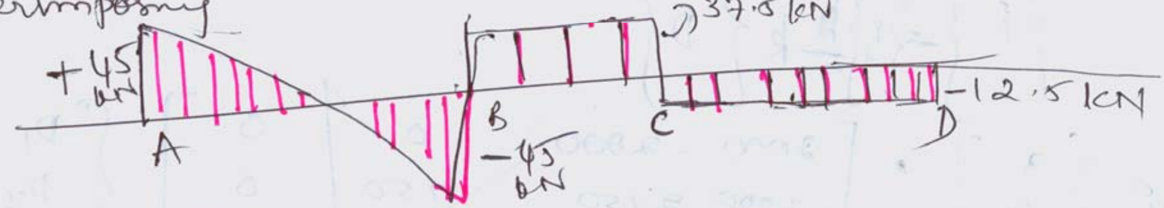


$\Sigma F_x = 0$ is satisfied

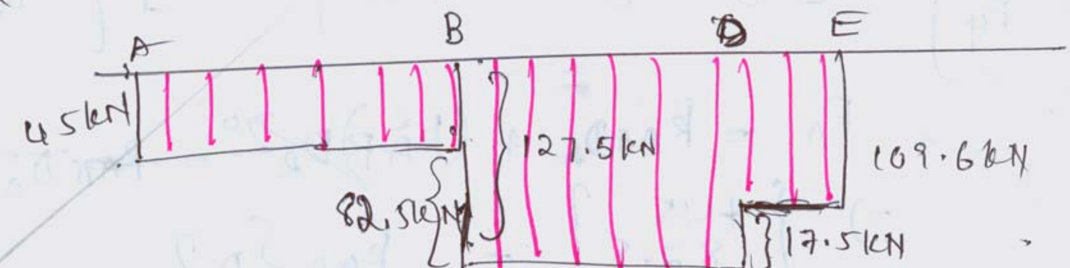
∴ The solution is correct



Superimposing



and



Axial force diagrams:

(ii) Displacement Transformation Matrix T^D

$$D^* = T^D D$$

$$T^D = \begin{array}{cc|cccc} & & ① & ② & ③ & ④ \\ ① & 1^* & 1 & 0 & 0 & 0 \\ & 2^* & 0 & 1 & 0 & 0 \\ ② & 1^* & 0 & 1 & 0 & 0 \\ & 2^* & 0 & 0 & 1 & 0 \\ ③ & 1^* & 0 & 0 & 1 & 0 \\ & 2^* & 0 & 0 & 0 & 1 \end{array}$$

$$k = T_D^T \, k_* \, T_D$$

where $k_* = \begin{bmatrix} k^1_* & 0 & 0 \\ 0 & k^2_* & 0 \\ 0 & 0 & k^3_* \end{bmatrix}$

(from previous problem

$$\Rightarrow \quad k = \begin{bmatrix} 2000 & -2000 & 0 & 0 \\ -2000 & 2750 & -750 & 0 \\ 0 & -750 & 8750 & -8000 \\ 0 & 0 & -8000 & 8000 \end{bmatrix} \text{ kN/m.}$$

$$\{F - F_f\} = k D.$$

$$F_{*f} = \begin{array}{cc} ① & 1^* \\ & 2^* \\ ② & 1^* \\ & 2^* \end{array} \begin{Bmatrix} -45 \\ -45 \\ -37.5 \\ -12.5 \end{Bmatrix} \text{ kN} \qquad F_f = T_D^T F_{*f}$$

$$F_f = \begin{Bmatrix} -45 \\ -82.5 \\ -12.5 \\ 0 \end{Bmatrix} \text{kN}$$

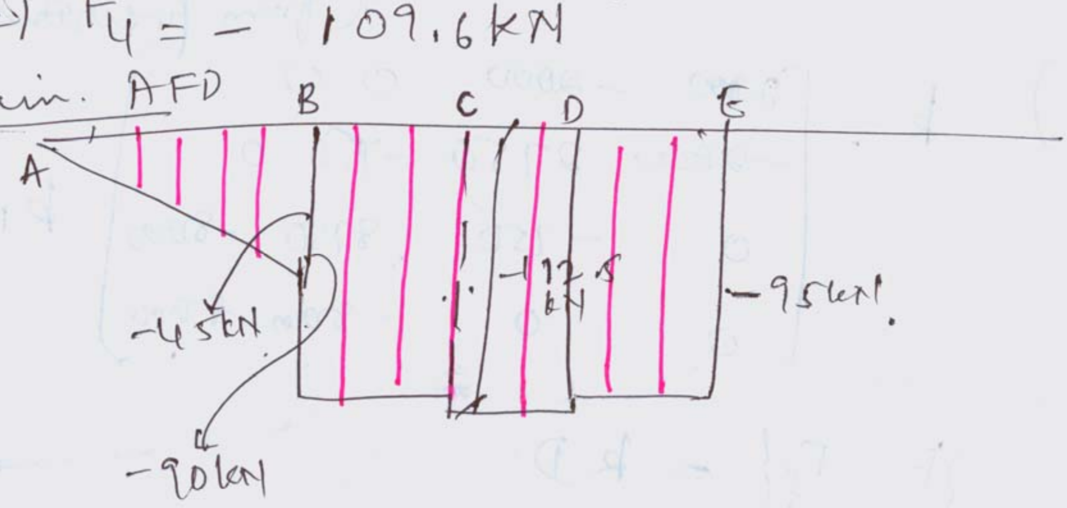
$$F = \begin{Bmatrix} 0 \\ 0 \\ -30 \\ F_4 \end{Bmatrix}$$

$$F - F_f = k\,D \quad (\text{from similar to (i) Part A})$$

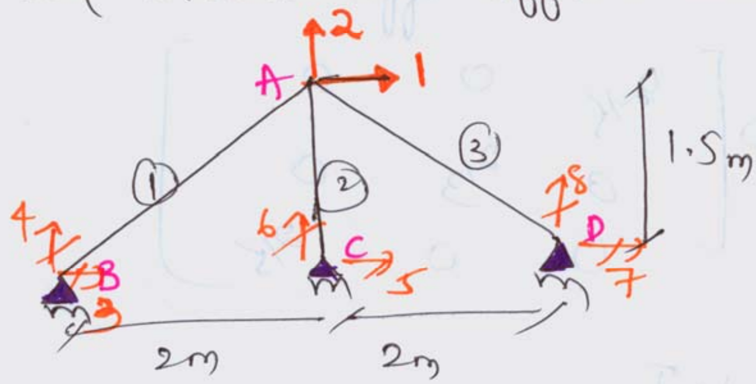
$$\Rightarrow \begin{Bmatrix} D_A \end{Bmatrix} = \begin{Bmatrix} 206.2\ \text{mm} \\ 183.7\ \text{mm} \\ 13.7\ \text{mm} \end{Bmatrix}$$

$$\Rightarrow F_4 = -109.6\ \text{kN}$$

Again. AFD



Q.(2) Reduced Element Stiffness Method.



$EA = 3000\,kN$

Bar 2 : short by 5 mm $\quad e_0 = \begin{Bmatrix} 0 \\ -5 \\ 0 \end{Bmatrix}$ mm.

(initial displacement vector)

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 50 \\ -25 \end{Bmatrix} kN$$

S No.	Start Node	End Node	θ_x^i	$\cos\theta_x^i$	$\sin\theta_x^i$	$L_i = \frac{1.5}{\sin}$
1	B	A	$\tan^{-1} 3/4$	$\frac{4}{5}$	$3/5$	2.5 m
2	C	A	90°	0	1	1.5 m
3	D.	A	$\cos^{-1}(-\frac{4}{5})$	$-\frac{4}{5}$	$\frac{3}{5}$	2.5 m

$$T_D^1 = 1 \begin{array}{c} \begin{array}{cccc} ③ & ④ & ① & ② \end{array} \\ \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \end{array}$$

$T_D^i = [-\cos\theta \;\; -\sin\theta \;\; +\ldots$

$$T_D^2 = 1 \begin{array}{c} \begin{array}{cccc} ⑤ & ⑥ & ① & ② \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Reduced element stiffness Matrix :

$$k^*_* = \cancel{EA}\, 3000 \begin{bmatrix} \cancel{2}/5 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/5 \end{bmatrix}$$

$$k = T_D^T \, k_* T_D$$

$$k_* T_D = \underset{3\times3}{k_*} \left[\underset{3\times2}{T_{DA}} \,\middle|\, \underset{3\times6}{T_{DR}} \right]$$

$$T_D^T k_* T_D = \cancel{T_{DA}^T}\; T_D^T \left[k_* T_{DA} \,\middle|\, k_* T_{DR} \right]$$

$$= \begin{bmatrix} T_{DA}^T \\ T_{DR}^T \end{bmatrix} \left[k_* T_{DA} \,\middle|\, k_* T_{DR} \right]$$

$$= \left[\begin{array}{c|c} T_{DA}^T k_* T_{DA} & T_{DA}^T k_* T_{DR} \\ \hline T_{DR}^T k_* T_{DA} & T_{DR}^T k_* T_{DR} \end{array}\right]$$

$$= \left[\begin{array}{c|c} k_{AA} & k_{AR} \\ \hline k_{RA} & k_{RR} \end{array}\right]$$

$$\Rightarrow k_{AA} = T_{DA}^T k_* T_{DA}$$

$$T_D = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & -4 & -3 & 0 & 0 & 0 & 0 \end{array}$$

$$[T_{DA}]_{3\times2} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ 0 & 1 \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\Rightarrow k_{AA} = T_{D*}^{T}\, k_{*}\, T_{DA}$$

$$= \begin{bmatrix} 1536 & 0 \\ 0 & 2864 \end{bmatrix} \text{ kN/m}$$

$$= 3000 \begin{bmatrix} 0.512 & 0 \\ 0 & 0.955 \end{bmatrix} \text{ kN/m}$$

Input Load Data :

Acting $\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 50 \\ -25 \end{Bmatrix}$ kN.

Bar 2 is too short by 5 mm, So

$$F_{f*}^{2} = -\frac{EA}{1.5} \times (-5\times10^{-3}) = \frac{0.01}{3} \times 3000$$

$$= 10 \text{ kN}$$

$$F_f^* = \begin{Bmatrix} 0 \\ +10 \\ 0 \end{Bmatrix} \text{ kN} \Rightarrow F_f^T = \{ 0 \ \underline{10} \ 0 \ 0 \ 0 \ \underline{-10} \ 0 \ 0 \} \text{ kN}$$

Using $F_f^T = T_D^T F_f^*$

Equilibrium

$$F - F_f = k_s D$$

$$\begin{Bmatrix} F_A - F_{fA} \\ F_R - F_{fR} \end{Bmatrix} = \begin{bmatrix} k_{AA} & k_{AR} \\ k_{RA} & k_{RR} \end{bmatrix} \begin{bmatrix} D_A \\ D_R = 0 \end{bmatrix}$$

$$\Rightarrow \quad D_A = [k_{AA}]^{-1} \{F_A - F_{fA}\} \qquad F_{fA} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \text{ kN}$$

$$= \begin{bmatrix} 1536 & 0 \\ 0 & 2684 \end{bmatrix}^{-1} \begin{Bmatrix} 50 \\ -25 - 10 = -35 \end{Bmatrix}$$

$$\{D_A\} = \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} 0.0325 \text{ m} \\ -0.013 \text{ m} \end{Bmatrix} = \boxed{\begin{Bmatrix} 32.5 \text{ mm} \\ -13.0 \text{ mm} \end{Bmatrix} \text{ Ans.}}$$

Bar forces $F_* = f_* \cancel{D_* \, [T_F]\{F_A\}}$

$$= \cancel{f_* \, T F'}$$

$$F_* = k_* D_*$$

$$= k_* T_D D$$

$$= 3000 \begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} T_D \end{bmatrix} D \qquad \cancel{-\tfrac{4}{5} \;\; -\tfrac{3}{5} \;\; \tfrac{4}{5} \;\; \tfrac{3}{5}}$$

$$= k_* \{T_{DA} \mid T_{DR}\} \begin{Bmatrix} D_A \\ D_R \end{Bmatrix}$$

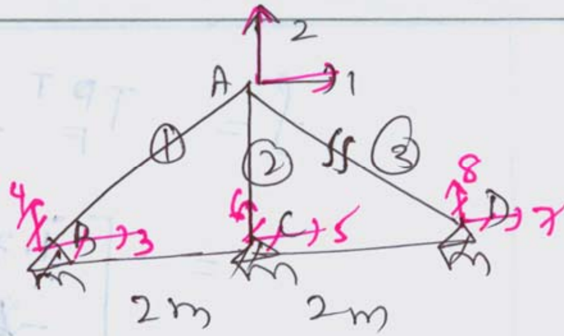
$$= k_* [T_{DA} D_A + T_{DR} \cancel{D_R}^{0}]$$

$$= k_* T_{DA} D_A$$

Member

$$F^A = 1 * [\; 21.84$$

Q-③



Bar ③ is redundant.

$$D_{initial\,*} = \begin{Bmatrix} 0 \\ -5\,mm \\ 0 \end{Bmatrix}$$

Load Input $= \{F_A\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} +50 \\ -25 \end{Bmatrix}$ kN

Flexibility Matrix

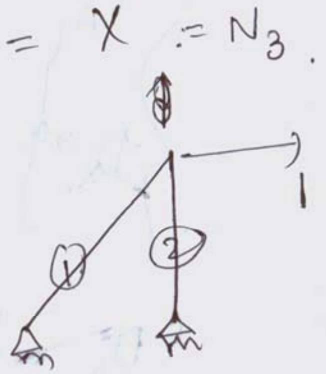
$$f_* = \begin{bmatrix} 5/2EA & 0 & 0 \\ 0 & 3/2EA & 0 \\ 0 & 0 & \frac{5}{2EA} \end{bmatrix} \,(kN/m)^{-1}$$

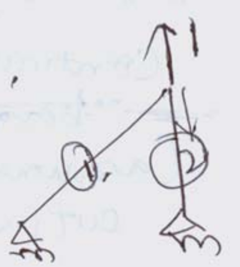
$$f_* = 10^{-4}\begin{bmatrix} 8.33 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8.33 \end{bmatrix} \,(kN/m)^{-1}$$

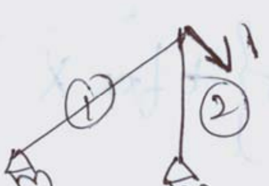
Bar ③ : Redundant $\Rightarrow F_X = F_3 = X = N_3$.

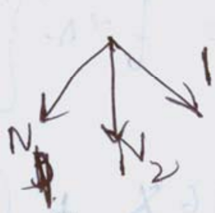
$F_* = T_F F$

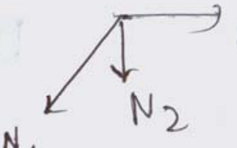
$$T_F = \begin{array}{c|ccc} & F_1=1 & F_2=1 & F_3=1 \\ \hline 1* & 1.25 & 0 & 1 \\ 2* & -0.9375 & +1 & -1.2 \\ 3* & 0 & 0 & 1 \end{array}$$











$N_1 \times \frac{3}{5} + N_2 = 0$

$N_1 = 1$

$N_1\,4 \ldots 1$

$$f = T_F^{T} \, f^{*} \, T_F$$

$$= \left[\begin{array}{cc|c} 1.741 & \frac{-4.69}{10} & 1.603 \\ \frac{-4.69}{10} & \frac{5}{10} & -6/10 \\ \hline 1.603 & -\frac{6}{10} & 2.386 \end{array}\right] \times 10^{-3} \quad (kN/m)^{-1}$$

Compatibility, :

$$f(F - F_{f*}) = \{D\} - \{D_{initial}\} \qquad F_{fA} = 0 \text{ here}.$$

$$D = D_{initial} + [f]\{F\}$$

$$\begin{Bmatrix} D_A \\ D_X \end{Bmatrix} = \begin{Bmatrix} D_{initial,A} \\ D_{initial,X} \end{Bmatrix} + \begin{bmatrix} f_{AA\,(2\times2)} & f_{AX} \\ f_{XA} & f_{XX\,(1\times1)} \end{bmatrix} \begin{Bmatrix} 50 \\ -25 \\ X \end{Bmatrix}$$

$$\begin{Bmatrix} D_A \\ \\ D_X \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5mm \\ 0 \end{Bmatrix} + [f]\{F\}$$

($D_{initial} = T_F^T \, D_{initial,x}$ Contragradient Principle. $= -5$ mm at (2))

$$D_A = \begin{Bmatrix} 0 \\ -5mm \end{Bmatrix} + f_{AA} \begin{Bmatrix} 50 \\ -25 \end{Bmatrix} + f_{AX} X$$

$D_X = 0$ for continuity of bar. (assuming cut in bar)

$$D_X = 0 = f_{XA} \begin{Bmatrix} 50 \\ -25 \end{Bmatrix} + f_{XX} X$$

$$\Rightarrow 2.386X + \left(1.603 \times 50 + 25 \times \frac{6}{10}\right) = 0.$$

$$X = 39.878 \approx 40 \, kN$$

$$D_A = \begin{Bmatrix} 0 \\ -5\,mm \end{Bmatrix} + f_{AA} \begin{Bmatrix} 50 \\ -25 \end{Bmatrix} - 40 \begin{Bmatrix} 1.603 \\ -0.6 \end{Bmatrix} \times 10^{-3}$$

$$= \begin{Bmatrix} 0 \\ -5 \end{Bmatrix} \times 10^{-3} + \begin{Bmatrix} \left(1.741 \times 50 + \frac{4.69}{10} \times 25\right) \\ + 50 \times \frac{4.69}{10} - 25 \times \frac{5}{10} \end{Bmatrix} 10^{-3} - 10^{-3} \begin{Bmatrix} 40 \times 1.603 \\ -40 \times 0.6 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -5\,mm \end{Bmatrix} + \begin{Bmatrix} 98.775\,mm \\ -35.95\,mm \end{Bmatrix} - \begin{Bmatrix} 64.12\,mm \\ -24\,mm \end{Bmatrix}$$

$$\boxed{D_A = \begin{Bmatrix} 34.655\,mm \\ -16.95\,mm \end{Bmatrix}}$$ Ans.

$$F^* = T_F \begin{Bmatrix} F \end{Bmatrix} = \begin{Bmatrix} 1.25 \times 50 + 1 \times (-40) \\ -0.9375 \times 50 + 1 \times (-25) - 1.2 \times (-40) \\ -40 \end{Bmatrix}$$

$$\boxed{F^* = \begin{Bmatrix} 22.5 \\ -23.875 \\ -40 \end{Bmatrix} kN}$$ Ans.

(The slight change in the answers is due to rounding