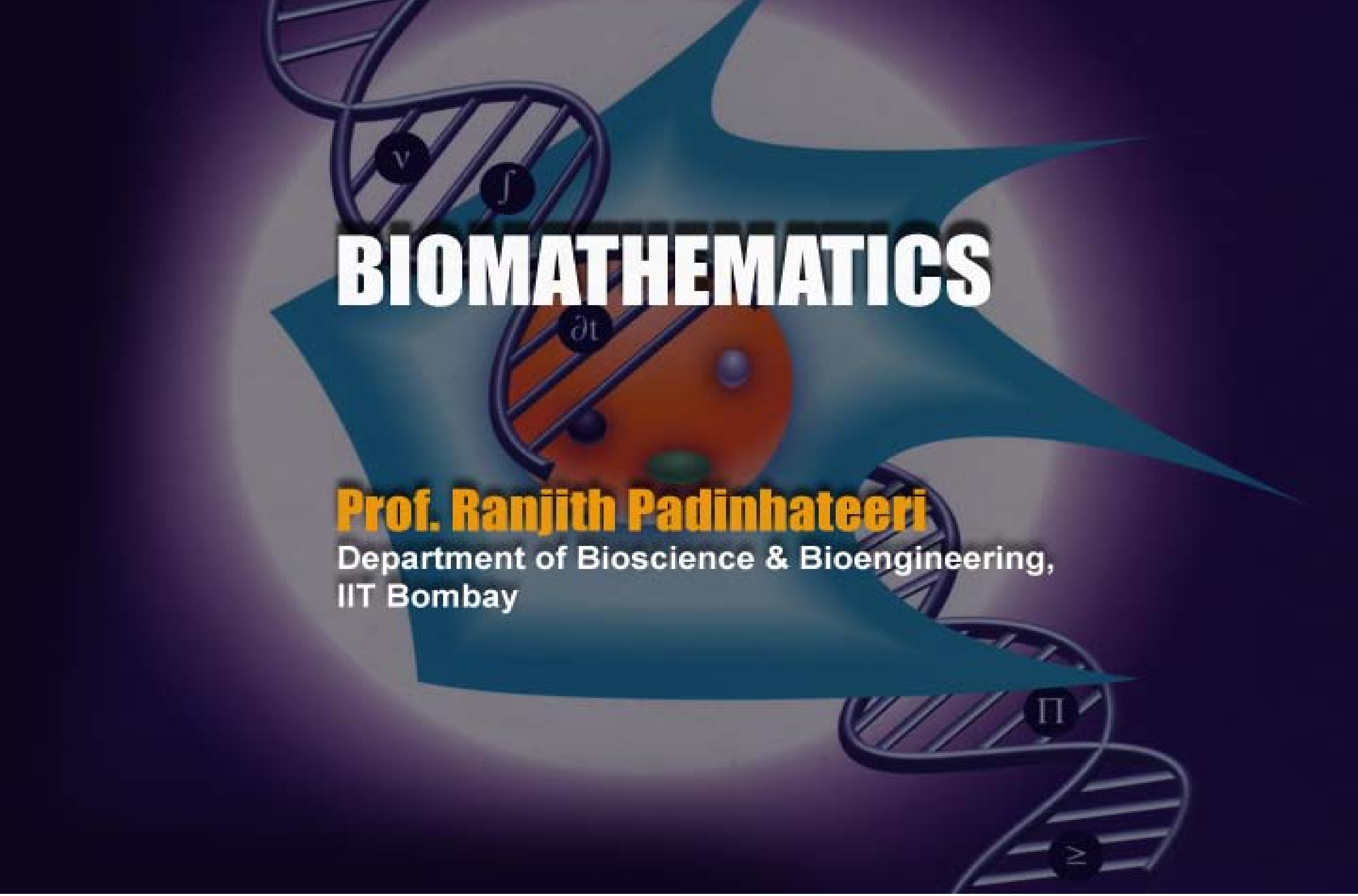
BIOMATHEMATICS
Prof. Ranjith Padinhateeri
Department of Bioscience & Bioengineering,
IIT Bombay

Lecture 28

Fourier Series

Fourier series

A period function f(x) can be written as sums of sines and cosines

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

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$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(nx) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Some properties of sin and cos

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0$$

Example of a Fourier series

$$f(x) = 0, \quad \text{for} - \pi < x < 0$$

$$f(x) = h, \quad \text{for } 0 < x < \pi$$

=> Square wave

Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} h dx \quad = h$$

Fourier coefficients

$$a_n = \frac{1}{\pi}\int_0^{\pi} h\ cos(nx)dx = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Fourier coefficients

$$b_n = \frac{1}{\pi}\int_0^{\pi} h\ sin(nx)dx$$

$$= \frac{h}{n\pi}(1 - cos\ n\pi)$$

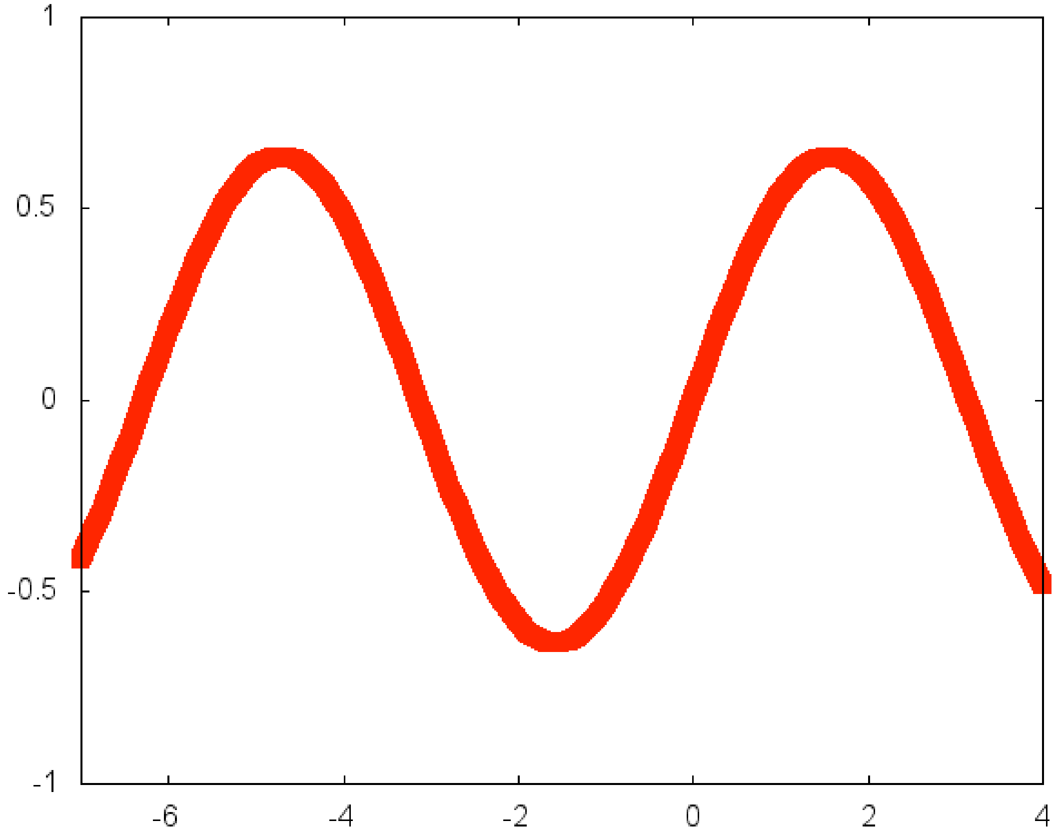
$$b_n = \frac{2h}{n\pi}, \quad \text{for } n \text{ odd},$$

$$b_n = 0, \qquad \text{for } n \text{ even},$$

The series

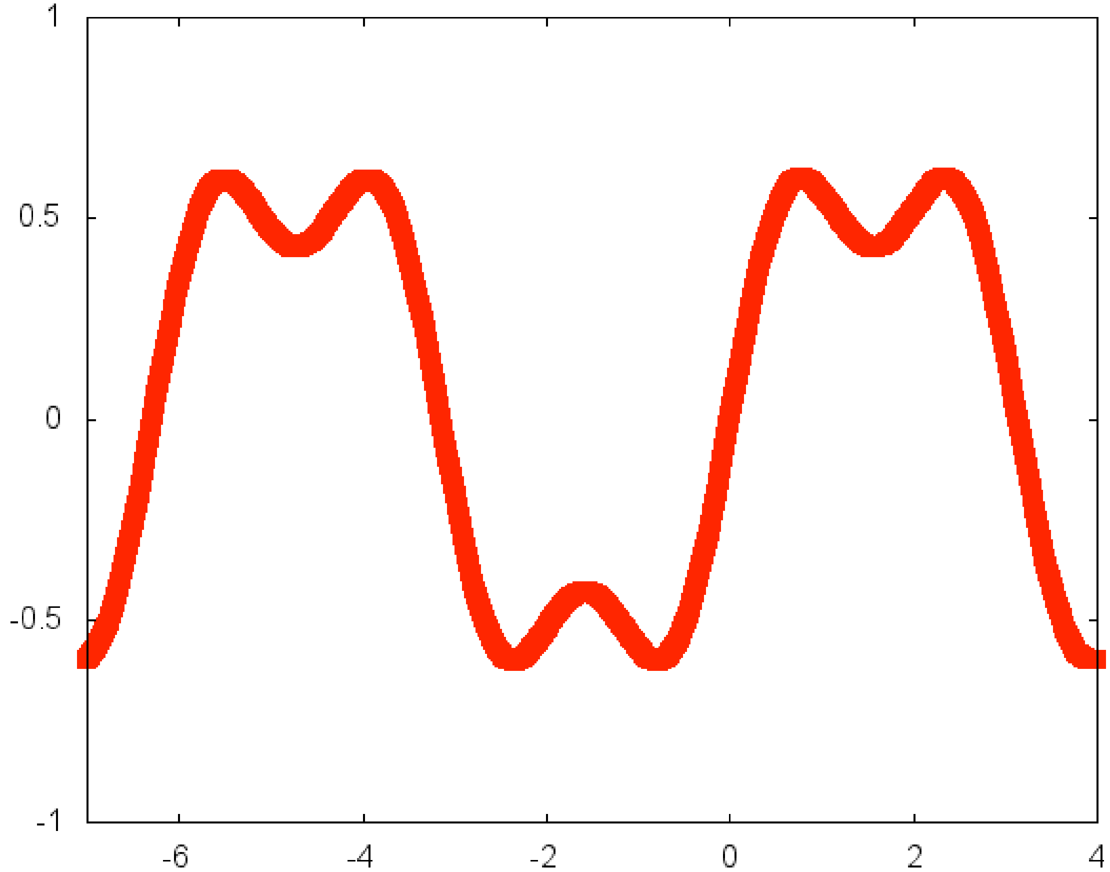
$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{sin\ x}{1} + \frac{sin\ 3x}{3} + \frac{sin\ 5x}{5} + ...\right)$$

1^{st} term



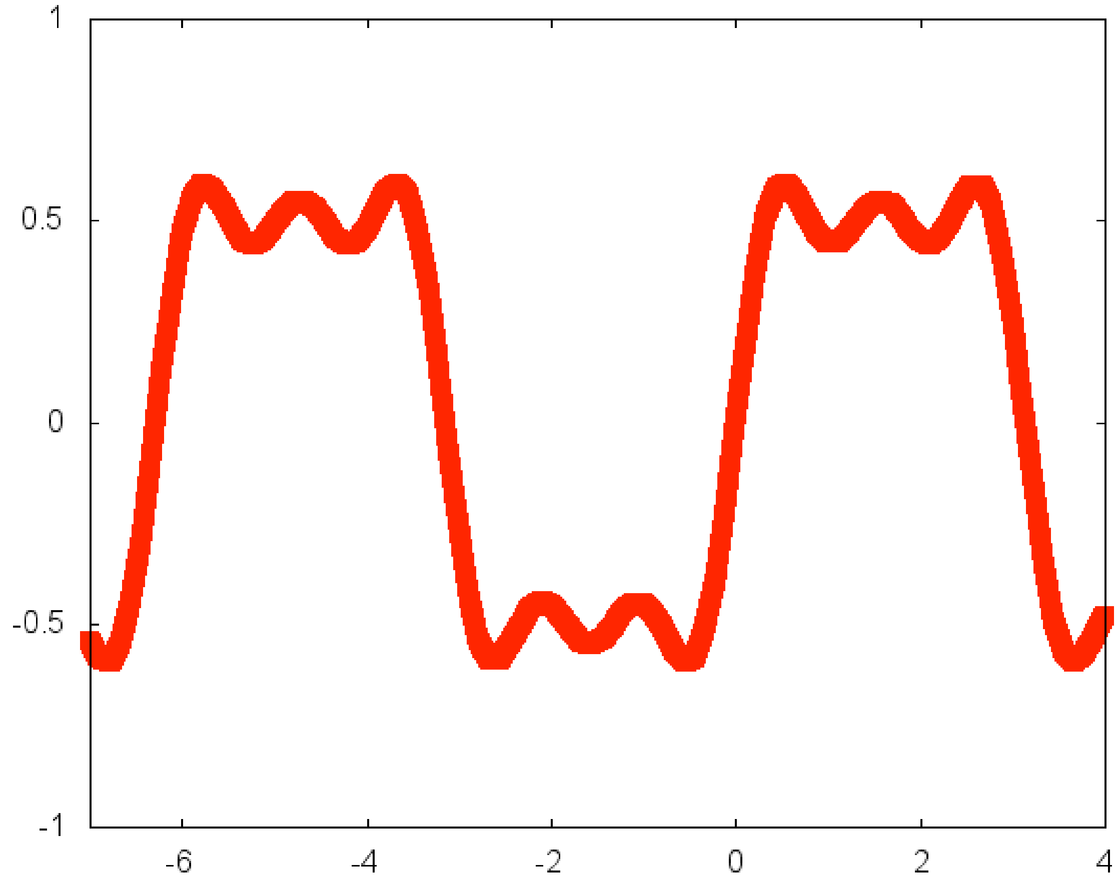
$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{sin\ x}{1}\right)$$

2 terms



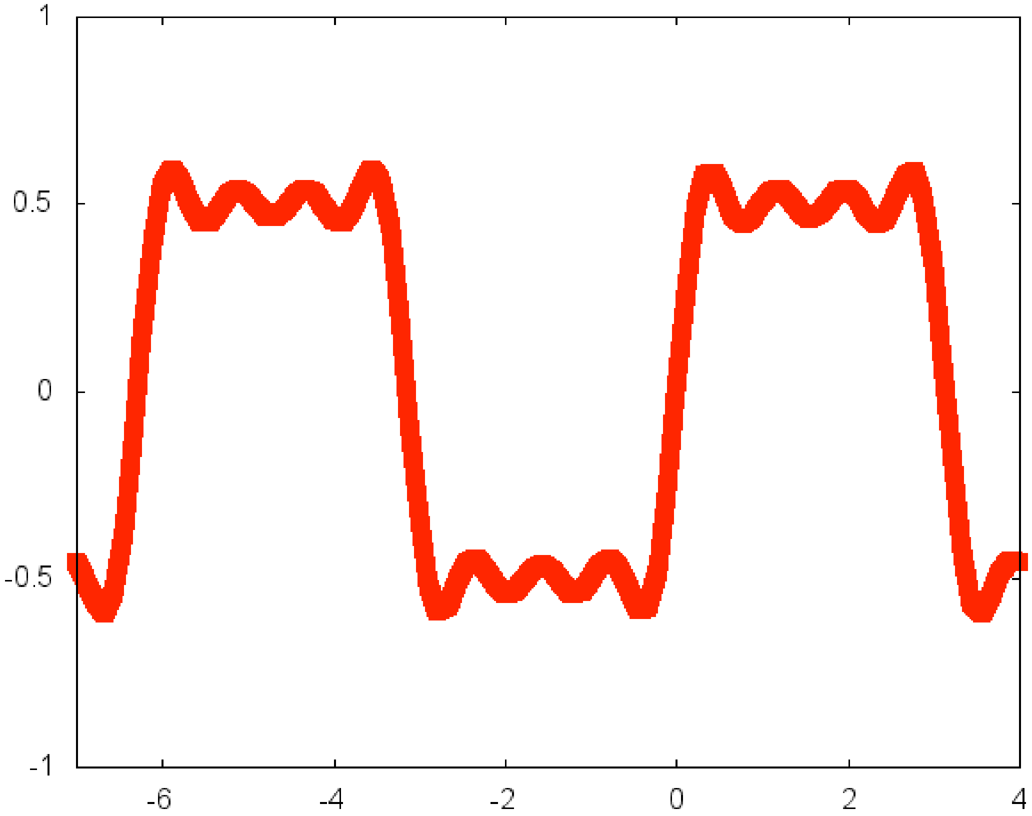
$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{sin\ x}{1} + \frac{sin\ 3x}{3}\right)$$

3 terms



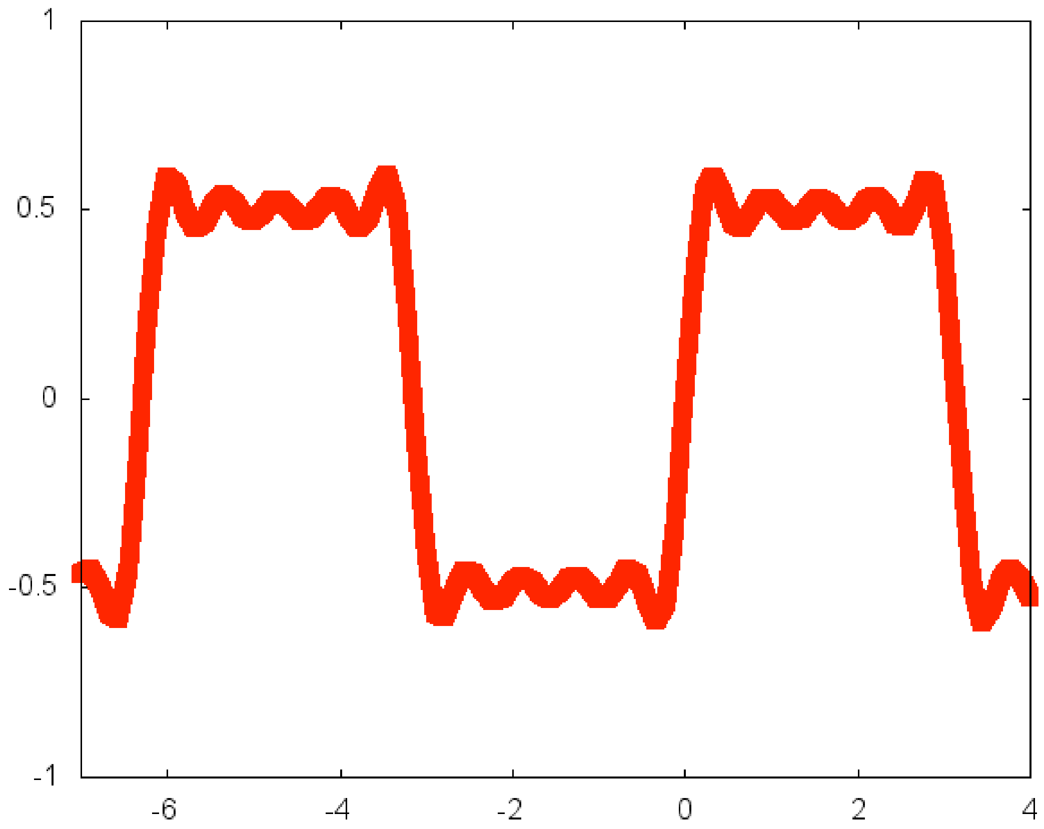
$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{sin\ x}{1} + \frac{sin\ 3x}{3} + \frac{sin\ 5x}{5}\right)$$

4 terms



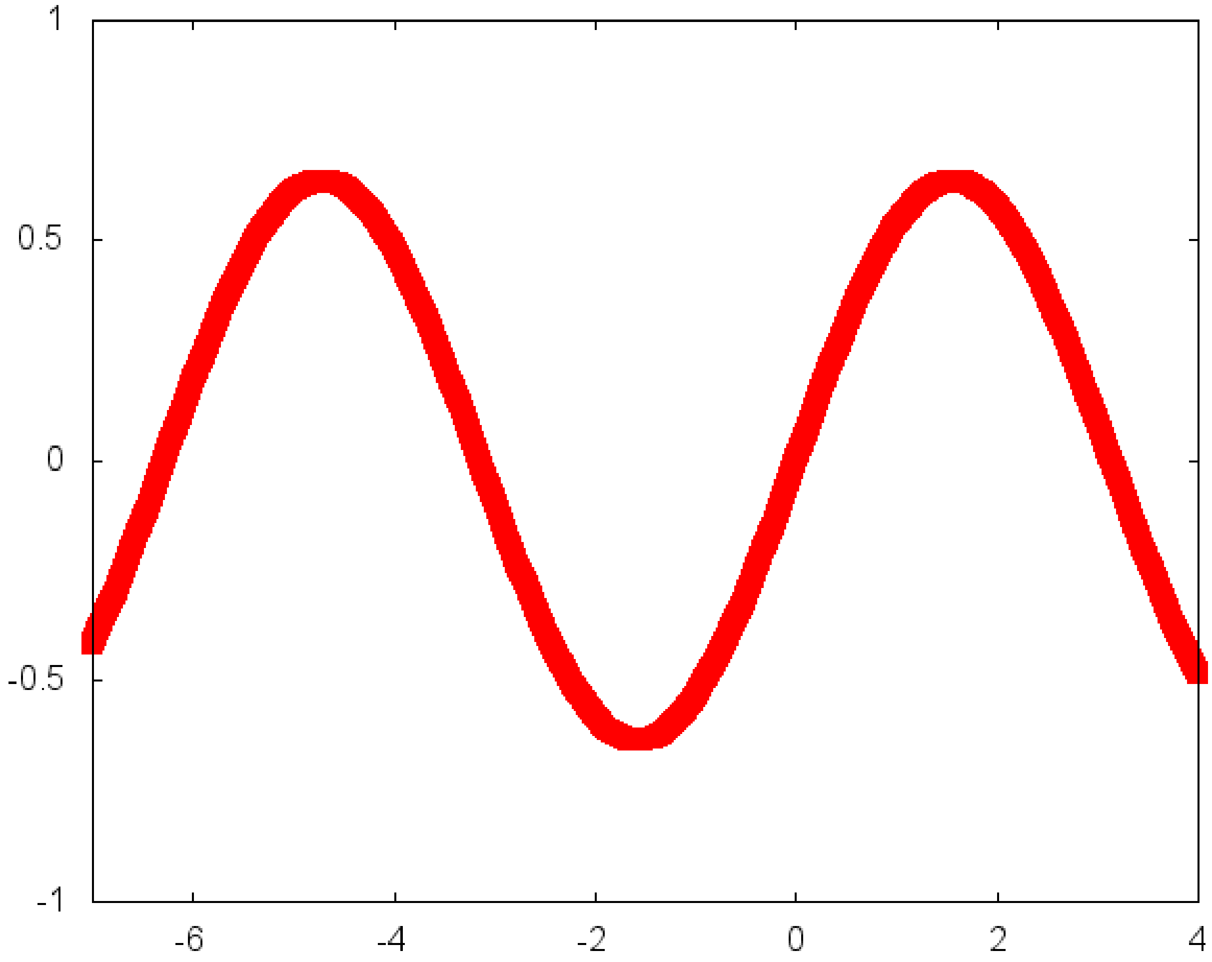
$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{sin\ x}{1} + \frac{sin\ 3x}{3} + \frac{sin\ 5x}{5} + \frac{sin\ 7x}{7}\right)$$

5 terms



$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{sin\ x}{1} + \frac{sin\ 3x}{3} + \frac{sin\ 5x}{5} + \frac{sin\ 7x}{7} + \frac{sin\ 9x}{9}\right)$$

Square wave



Exponential Fourier series

Euler's formula,

$$e^{inx} = cos(nx) + i\ sin(nx)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Exponential Fourier series

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Relation among the coefficients

$$a_n = c_n + c_{-n} \quad \text{for } n = 0, 1, 2, \ldots$$

$$b_n = i(c_n - c_{-n}) \quad \text{for } n = 1, 2, \ldots$$

$$c_n = \frac{1}{2}(a_n - ib_n), \quad \text{for } n > 0$$
$$c_n = \frac{1}{2}a_0, \quad \text{for } n = 0$$
$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}), \quad \text{for } n < 0$$