Lecture – 15

Review of Matrix Theory – III

Dr. Radhakant Padhi

Asst. Professor

Dept. of Aerospace Engineering

Indian Institute of Science - Bangalore

Matrix

An $m \times n$ matrix is a rectangular or square array of elements with m rows and n columns.

$$\text{Eg:} \quad A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

For each subscipt, a_{ij}, i = the row, and j = the column.

If $m = n$, then the matrix is said to be a "square matrix".

Vector

If a matrix has just one row, it is called a "row vector"

Eg. $\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \end{bmatrix}$

If a matrix has just one column, it is called a "column vector"

Eg. $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$

Null Matrix / Diagonal Matrix

Null matrix : A matrix with all zero elements

$$\text{Eg.} \quad A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Diagonal Matrix : A square matrix with all off diagonal elments being zero

$$\text{Eg.} \quad A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Linear Dependence/Independence

A set of n column vectors $s_1, s_2, \cdots, s_n$, is said to be "linearly dependent" if there exist constants $\alpha_1, \alpha_2, \cdots, \alpha_n$, not all zero, such that

$$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n = 0$$

If the above equation holds only when $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the vectors are "linearly independent".

Example: Linear Dependence/Independence

Question : Are the vectors $a_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $a_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$,

$a_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ linearly independent?

Answer : We must test whether

$xa_1 + ya_2 + za_3 = 0$ admits any non-trivial solution.

The determinant of the coefficient is

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \neq 0$$

Hence the vectors are linearly independent.

Rank of a Matrix

The *rank* of a matrix $A_{m\times n}$ is equal to the number of linearly independent rows or columns. The rank can be found by finding the higest-order square submatrix that is non-singular.

Eg. $A = \begin{bmatrix} 1 & -5 & 2 \\ 4 & 7 & -5 \\ -3 & 15 & -6 \end{bmatrix}$

Here $|A| = 0$. So, the A matrix is singular and hence $rank(A) \neq 3$

Choosing the sub-matrix $B = \begin{bmatrix} 1 & -5 \\ 4 & 7 \end{bmatrix}$, $|B| = 27 \neq 0$

Hence $rank(A) = 2$

Matrix Operations

Addition

The sum of two matrices, written A + B = C, is defined

by $a_{ij} + b_{ij} = c_{ij}$ Eg. $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -1 & 8 \end{bmatrix}$

Subtraction

The difference between two matrices, written $A - B = C$, is defined

by $a_{ij} - b_{ij} = c_{ij}$ $Eg.$ $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 7 & 2 \end{bmatrix}$

Matrix Operations

Multiplication

The product of two matrices, written $AB = C$, is defined by $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{jk}$

Eg. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}; \; B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

Schur / Hadamard / Direct product

$A \bullet B = \left[a_{ij} \bullet b_{ij} \right]$, if A & B have some dimension

Matrix Identities

Commutative Law $A + B = B + A$

$$AB \neq BA$$

Associative Law $\mathrm{A} + (B + C) = (A + B) + C$

$$\mathrm{A}(BC) = (AB)C$$

Transpose of sum $(A + B)^T = A^T + B^T$

Transpose of product $(AB)^T = B^T A^T$

Determinant identities $\det A^T = \det A$

$$\det AB = \det A \det B$$

$$\det AB = \det BA$$

Trace

Trace of $A_{n\times n}$

$$T_r(\mathrm{A}) = \sum_{i=1}^{n} a_{ij} \quad : \text{ Sum of the diagonal elements}$$

Eg $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \; Tr(\mathrm{A}) = 1 + 4 = 5$

Example: Eigenvalues/Eigenvectors

Question : Find eigenvalues/eigenvecotrs of $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

Solution :

Eigenvalues: The characteristic equation is given by

$$\det\left(A - \lambda I_2\right) = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$$

Hence the eigenvalues are $\lambda_1 = -1, \lambda_2 = 4$

Example: Eigenvalues/Eigenvectors

Eigenvectors:

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$. Then the equation $\left(A - \lambda_1 I_2\right) X_1 = 0$ leads to:

$$\begin{bmatrix} 1-4 & 3 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} -3a_1 + 3b_1 = 0 \\ 2a_1 - 2b_1 = 0 \end{Bmatrix} \Rightarrow \left(a_1 = b_1\right)$$

The solutions can be expressed as $a_1 = b_1 = \alpha$

for any α. If we put $\alpha = 1$, then an eigenvector is

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example: Eigenvalues/Eigenvectors

Similarly $\left(A-\lambda_2 I_2\right)X_2=0$ implies

$$\begin{bmatrix} 1+1 & 3 \\ 2 & 2+1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}=0, \text{ or } \begin{Bmatrix} 2a_2+3b_2=0 \\ 2a_2+3b_2=0 \end{Bmatrix}$$

The eigenvectors for this case are

$$X_2=\begin{bmatrix} \beta \\ -\dfrac{2}{3}\beta \end{bmatrix}$$

for any nonzero β. Assuming $\beta=3$ gives the eigenvector

$$X_2=\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Example: Eigenvalues/Eigenvectors

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Additional Properties of Eigenvalues/Eigenvectors

$$\left|A\right| = \lambda_1 \cdots \lambda_n$$

$$Tr\left(A\right) = \lambda_1 + \cdots + \lambda_n$$

If $\det\left(\lambda I - A\right) = \lambda_n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

then $a_{n-1} = -Tr\left(A\right)$, $\quad a_0 = \left(-1\right)^n \left|A\right|$

If $A_{n\times n}$ is Real and Skew-Symmetric $\left(A = -A^T\right)$

$\left[\text{or Complex and Skew-Hermition } \left(A = -A^*\right)\right]$,

Then its eigenvalues are all pure imaginary

Additional Properties of Eigenvalues/Eigenvectors

If $\mathrm{A}_{n\times n}$ is Real and Symmetric $\left[\text{or Complex and Hermition}\right]$, then its eigenvalues are all real.

If A is a symmetrical matrix, then the eigenvectors associated with two distirct eigenvalues are orthogonal

If $A_{n\times n}$ is a symmetric matrix, then

$$\lambda_{\min} < R(X) < \lambda_{\max} \quad \text{, where, } R(X) \triangleq \left(\frac{X^T \mathrm{A} X}{X^T X} \right), \ X \neq 0$$

Proof

Proof

Proof

Proof

Generalized Eigenvectors

Suppose $A_{3\times3}$ has eigenvalues $\lambda_1, \lambda_2, \lambda_2$

Let $A_\varepsilon \triangleq A + \varepsilon B$, $|\varepsilon| << 1$

such that A_ε has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$

where $\lambda_3 = \lambda_2 + \delta$, $|\delta| << 1$

and the corresponding eigenvectors $v_1, v_2, v_3 = v_2 + \delta w$

Then as $\varepsilon \to 0$, the following equations are satisfied:

(1) $(A - \lambda_i I) v_i = 0 \qquad i = 1, 2$

(2) $(A - \lambda_2 I) w = v_2$

Note that $\{v_1, v_2, w\}$ are linearly independent

and w is called a "generalited eigenvector" of A.

Quadratic Forms

Suppose $X = \left[x_1, x_2, \cdots, x_n \right]^T$. Then any polynomial function the elements in which every term is of degree two is known as a quadratic form. Thus, if $n = 3$, then

$$x_1^2 + 8x_1x_2 + x_2{}^2 + 6x_2x_3 + x_3{}^2$$

is an example of a quadratic form.

Quadratic forms can always be expressed as a matrix product of the form $X^T AX$

Utility: Optimization theory, Optimal control theory etc.

Example: Quadratic Form

The example above can be written as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In this representation, A is required to be symmetric matrix,

Nonsymmmetric representations are possible with

$$\text{Eg: } A = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix}$$

but the symmetric form is "standard"

Example: Singular Value Decomposition

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\lambda_{1,2,3}\left(A^T A\right) = 4, 0, 0$$

$$X_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\left(for \quad \lambda = 4 \right) \quad \left(for \quad \lambda = 0 \right) \quad \left(for \quad \lambda = 0 \right)$$

Example: Singular Value Decomposition

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \\ \sqrt{\lambda_3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \left(\text{Note: The values are ordered}\right)$$

$$Y_1 = \frac{1}{\sigma_1}\left(AX_1\right) = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Next, find $Y_2 = \begin{bmatrix} a & b \end{bmatrix}^{\mathrm{T}}$ such that Y_1 and Y_2 will be orthonormal

i.e. $\left\langle Y_1, Y_2 \right\rangle = Y_1^T Y_2 = a - b = 0$ and $\left\langle Y_2, Y_2 \right\rangle = Y_2^T Y_2 = a^2 + b^2 = 1$

Example: Singular Value Decomposition

Solution: $a = b = 1/\sqrt{2}.$ Hence, $Y_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} Y_1 & Y_2 \\ \downarrow & \downarrow \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Verify: \quad PDQ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} = A$$

Derivative of $A^{-1}(t)$

Let $A^{-1}(t) = B(t)$; Then $A(t)B(t) = I$

Hence $$\frac{d}{dt}\left(A(t)B(t)\right) = 0$$

$$A(t)\frac{dB(t)}{dt} + \frac{dA(t)}{dt}B(t) = 0$$

$$\frac{dB(t)}{dt} = -A^{-1}(t)\frac{dA(t)}{dt}B(t)$$

$$\boxed{\frac{dA^{-1}(t)}{dt} = -A^{-1}(t)\frac{dA(t)}{dt}A^{-1}(t)}$$

Practice Problems

1. Let $A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find eigenvalues/eigenvectors of A. In addition, find the eigenvalues/eigenvectors of A^{-1} , A^m $(m=3,4)$ and $I+2A+4A^2$ without computing these matrices.
2. Proove $(AB)^{-1} = B^{-1}A^{-1}$, if the indicated matrix inverses exist.
3. An $n\times n$ Hadamard matrix A has all elements that are ± 1 and satisfies $A^T A = nI$. Show that $\left|\det (A)\right| = n^{n/2}$
4. Show that the determinant of a negative definite $n\times n$ matrix is positive if n is even and negative if n is odd.
5. Let A and P be $n\times n$ matrices with P nonsingular. Show that $Tr(A) = Tr(P^{-1}AP)$

Practice Problems

6. Following the standard definitions,

   Show that for a fixed $X \in R^n \; \|X\|_p \to \|X\|_\infty$ as $p \to \infty$

7. Compute $\|A\|_1, \|A\|_2, \|A\|_\infty$ and $p(A)$ of the matrix in Problem-1

8. Find the singular value decomposition of

   $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Give all the steps. Verify the results using MATLAB.

9. Show that $\dfrac{\partial}{\partial X}\left(\dfrac{1}{2} X^T \; AX\right) = AX$

10. Show that if $F\left(f\left(X\right)\right) \in R^n, \; X \in R^n,$

    then $\left[\dfrac{\partial F}{\partial X}\right]_{p \times n} = \left[\dfrac{\partial F}{\partial f}\right]^T \left[\dfrac{\partial F}{\partial X}\right]_{m \times n}$

Thanks for the Attention....!