Lecture – 10

Conversion Between State Space and Transfer Function Representations in Linear Systems – II

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An Alternate First Companion Form
(Toeplitz first companion form)

$$H(s) = \left[\frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right] = \frac{y(s)}{u(s)}$$

i.e.

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y$$
$$= \quad b_0 u + b_1 \dot{u} + \cdots + b_{n-1} u^{(n-1)} + b_n u^{(n)}$$

Ref: B. Friedland,
Control System Design
Mc Graw Hill, 1986.

An Alternate First Companion Form
(Toeplitz first companion form)

Define the state variables $x_1, \cdots, x_n$ such that

$$y = x_1 + p_0 u$$

$$\dot{x}_1 = x_2 + p_1 u$$

$$\dot{x}_2 = x_3 + p_2 u$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n + p_{n-1} u$$

$$\dot{x}_n = -a_{n-1} x_n - \cdots - a_0 x_1 + p_n u$$

An Alternate First Companion Form
(Toeplitz first companion form)

From the above definition, we have

$$y = x_1 + p_0 u$$

$$\dot{y} = \dot{x}_1 + p_0 \dot{u} = \left(x_2 + p_1 u \right) + p_0 \dot{u}$$

$$\ddot{y} = \dot{x}_2 + p_1 \dot{u} + p_0 \ddot{u} = \left(x_3 + p_2 u \right) + p_1 \dot{u} + p_0 \ddot{u}$$

$$\vdots$$

$$y^{(n-1)} = x_n + p_{n-1} u + p_{n-2} \dot{u} + \cdots + p_1 u^{(n-2)} + p_0 u^{(n-1)}$$

$$y^{(n)} = -a_{n-1} x_n - a_{n-2} x_{n-1} - \cdots - a_0 x_1 + p_n u$$
$$+ p_{n-1} \dot{u} + p_{n-2} \ddot{u} + \cdots + p_1 u^{(n-1)} + p_0 u^{(n)}$$

An Alternate First Companion Form
(Toeplitz first companion form)

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y$$

$$= \left(p_n + a_{n-1} p_{n-1} + \cdots + a_1 p_1 + a_0 p_0 \right) u$$

$$+ \left(p_{n-1} + \cdots + a_2 p_1 + a_1 p_0 \right) \dot{u}$$

$$+ \cdots$$

$$+ \left(p_1 + a_{n-1} p_0 \right) u^{(n-1)}$$

$$+ \; p_0 u^{(n)}$$

$$= b_0 u + b_1 \dot{u} + \cdots + b_{n-1} u^{(n-1)} + b_n u^{(n)}$$

(from the TF)

An Alternate First Companion Form
(Toeplitz first companion form)

Equating the coefficients:

$$
\begin{aligned}
p_0 &= b_n \\
p_1 + a_{n-1} p_0 &= b_{n-1} \\
&\vdots \\
p_{n-1} + a_{n-1} p_{n-1} + \cdots + a_1 p_0 &= b_1 \\
p_n + a_{n-1} p_{n-1} + \cdots + a_0 p_0 &= b_0
\end{aligned}
$$

Now, we need to solve for $\left(p_0, p_1, \ldots, p_n \right)$

An Alternate First Companion Form
(Toeplitz first companion form)

In matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{n-1} & 1 & 0 & \cdots & 0 \\ a_{n-2} & a_{n-1} & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} & 1 \end{bmatrix}}_{\textbf{Toeplitz Matrix}} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix}$$

Note : Toeplitz matrix is a nonsinular matrix

$(\text{determinant} = 1 \neq 0)$

Hence, the solution always exists!

An Alternate First Companion Form
(Toeplitz first companion form)

State space form (from definition of state & output variables):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} X + \begin{bmatrix} p_0 \end{bmatrix} u$$

Note : The physical meaning of state variables in different companion froms are differnet.

Block Diagram for realization of Toeplitz first companion form

Alternate/Toeplitz first companion form: Some comments

- Toeplitz realization also requires ‘*n*’ integrators
- Extension of Toeplitz realization is straightforward to MISO systems
  - One need to solve *m*-system of linear equations, where *m* is the number of input (control) variables.
  - However, only *n* integrators are needed.

Second Companion Form
(Observable Canonical Form)

$$H(s) = \left[\frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right] = \frac{y(s)}{u(s)}$$

i.e.

$$\left(s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \right) y(s)$$
$$= \left(b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 \right) u(s)$$

Rearranging the terms:

$$s^n \left[y(s) - b_n u(s) \right] + s^{n-1} \left[a_{n-1} y(s) - b_{n-1} u(s) \right] + \cdots$$
$$\cdots + \left[a_0 y(s) - b_0 u(s) \right] = 0$$

Second Companion Form
(Observable Canonical Form)

Simplify:

$$\left[y(s)-b_n u(s)\right]=\frac{1}{s}\left[b_{n-1}u(s)-a_{n-1}y(s)\right]+\cdots+\frac{1}{s^n}\left[b_0 u(s)-a_0 y(s)\right]$$

Solve for *y(s)*:

$$y(s)=b_n u(s)+\frac{1}{s}\left[b_{n-1}u(s)-a_{n-1}y(s)\right]+\cdots+\frac{1}{s^n}\left[b_0 u(s)-a_0 y(s)\right]$$

$$=b_n u(s)+\underbrace{\frac{1}{s}\left[\left[b_{n-1}u(s)-a_{n-1}y(s)\right]+\underbrace{\frac{1}{s}\left[\left[b_{n-2}u(s)-a_{n-2}y(s)\right]\cdots+\right]}_{x_2(s)}\cdots\right]}_{x_1(s)}$$

Block Diagram Realization

Second Companion Form
(Observable Canonical Form)

The equations can be written as:

$$y = x_1 + b_n u$$

$$\dot{x}_1 = \left(x_2 - a_{n-1} y + b_{n-1} u \right) = -a_{n-1} x_1 + x_2 + \left(b_{n-1} - a_{n-1} b_n \right) u$$

$$\dot{x}_2 = \left(x_3 - a_{n-2} y + b_{n-2} u \right) = -a_{n-2} x_1 + x_3 + \left(b_{n-2} - a_{n-2} b_n \right) u$$

$$\vdots$$

$$\dot{x}_{n-1} = \left(x_n - a_1 y + b_1 u \right) = -a_1 x_1 + x_n + \left(b_1 - a_1 b_n \right) u$$

$$\dot{x}_n = b_0 u - a_0 y = -a_0 x_1 + \left(b_0 - a_0 b_n \right) u$$

Second Companion Form
(Observable Canonical Form)

In vector-matrix form, we can write it as

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ -a_1 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} b_{n-1} - a_{n-1} b_n \\ b_{n-2} - a_{n-2} b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{C} X + \underbrace{\begin{bmatrix} b_n \end{bmatrix}}_{D} u$$

Second Companion Form
(Observable Canonical Form)

(1) In second companion form, the coefficients of the denominator of the transfer function appear in a column, whereas in the first companion form they appear in a row.

(2) Extensions of second companion form to SIMO, MISO cases etc. are possible (but beyond the scope this course)

(3) Controllable canonical form is good for control design, whereas observable canonical form is good for observer/filter design

Comment on minimal realization (for MIMO systems)

We have seen realization of:

- **SIMO systems: *n* integrators**
- **MISO systems: *n* integrators**

Question: How about MIMO systems? Will it still be possible to realize it with *n* integrators?

Answer: No!

However, one way of realizing MIMO systems will be to use a number of structures (of either SIMO or MISO form) in parallel; *i.e. if* $U \in \mathbb{R}^m \text{ and } Y \in \mathbb{R}^p$

then it is always possible to realize such a MIMO system with no more than ***n x min (m, p)*** integrators.

Comment on minimal realization (for MIMO systems)

Question: How about fewer integrators?

Answer: This leads to the questions of "*minimal realization*"; a subject of considerable research during 1970s.

Why necessary? Because a non-minimal realization is either non-controllable or non-observable (or both). It may cause theoretical and computational problems too.

Solution: Several algorithms exist for finding a minimal realization and corresponding *A,B,C,D* matrices. However, these are beyond the scope of this course!

Further reading: T. Kailath, Linear Systems, Prentice Hall, 1980.

Thanks for the Attention...!