Lecture – 12

Overview of Flight Dynamics – II

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Six – DOF Model

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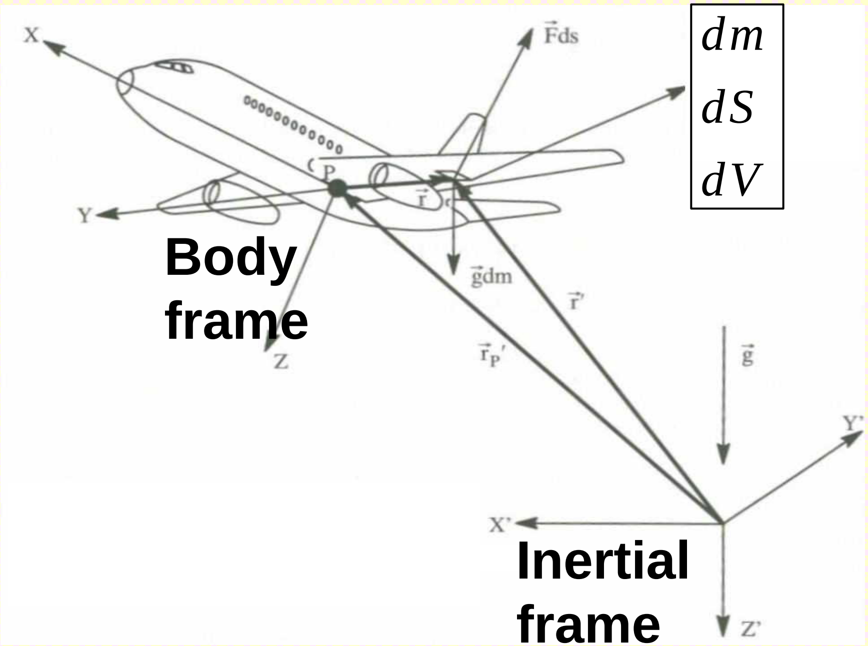
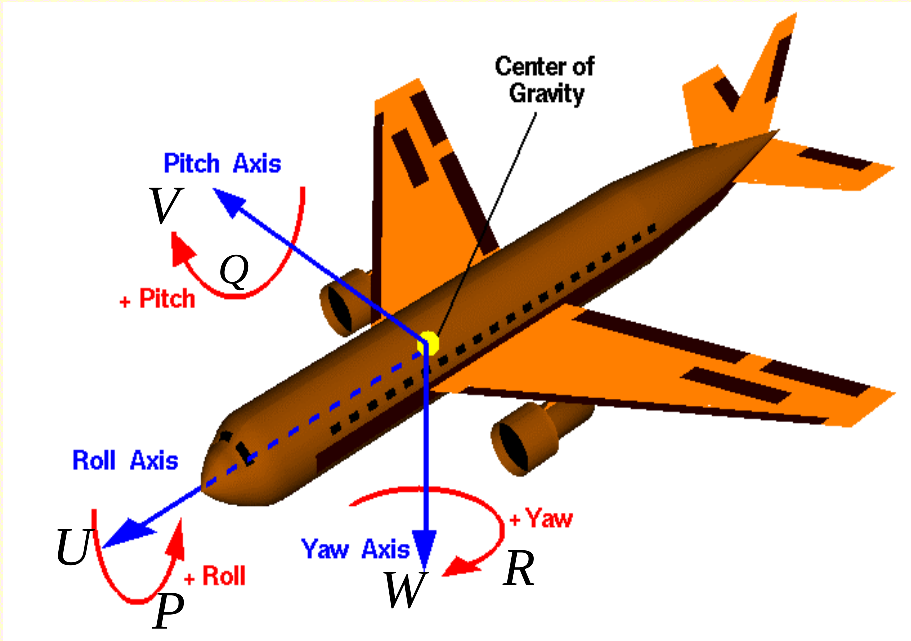
Reference Frames and Dynamic Variables





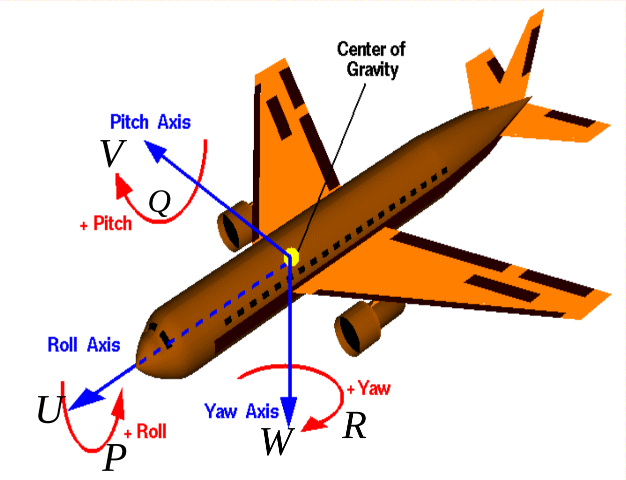
Dynamic (Force and Moment) Equations

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

$$\dot{U} = VR - WQ - g\sin\Theta + \frac{1}{m}\left(X + X_T\right)$$

$$\dot{V} = WP - UR + g\sin\Phi\cos\Theta + \frac{1}{m}\left(Y + Y_T\right)$$

$$\dot{W} = UQ - VP + g\cos\Phi\cos\Theta + \frac{1}{m}\left(Z + Z_T\right)$$



$$\dot{P} = c_1 QR + c_2 PQ + c_3\left(L + L_T\right) + c_4\left(N + N_T\right)$$

$$\dot{Q} = c_5 PR - c_6\left(P^2 - R^2\right) + c_7\left(M + M_T\right)$$

$$\dot{R} = c_8 PQ - c_2 QR + c_4\left(L + L_T\right) + c_9\left(N + N_T\right)$$

Kinematic Equations

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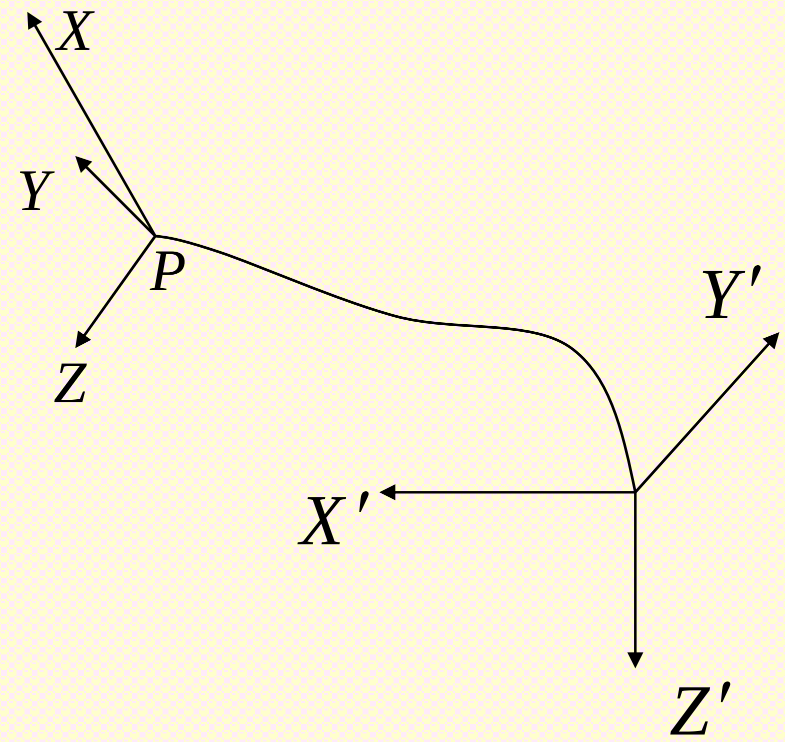
Orientation of Airplane wrt. Inertial Frame: Euler Angles

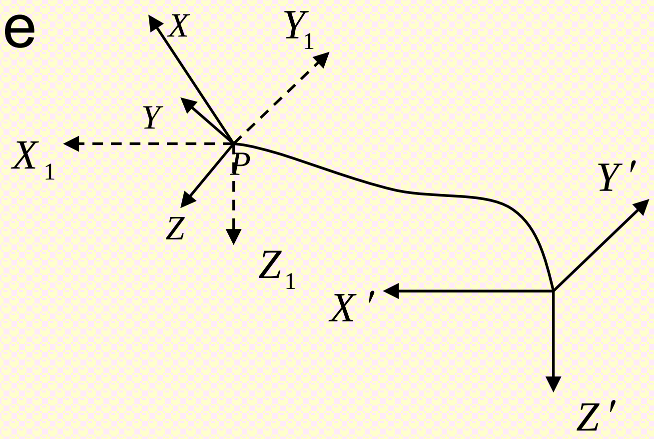
- Translate the inertial frame and make it coincide with the CG
- Make the sequential transformation of this frame so as to make it parallel to the body frame.
- Common sequence: Ψ, Θ, Φ

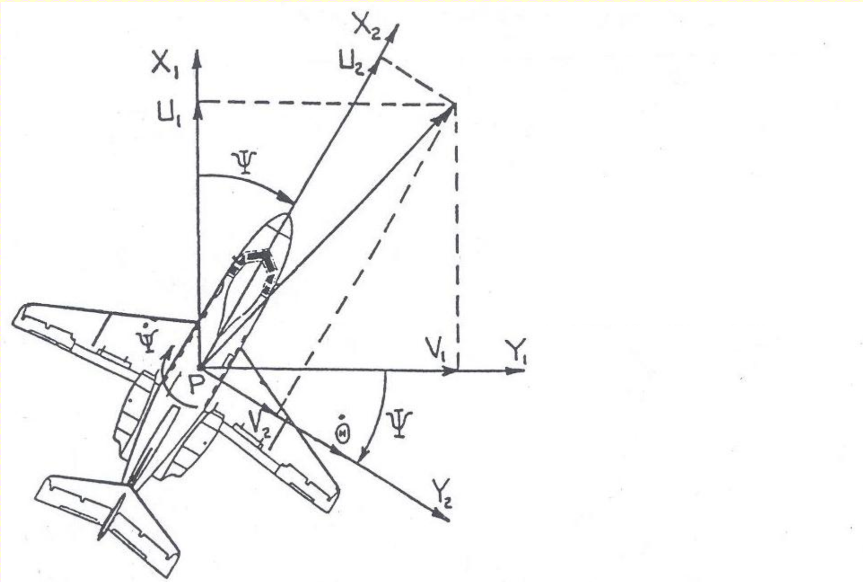


Euler Angles

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

Translate X'Y'Z' parallel to itself until its center coincides with the XYZ system. Rename X'Y'Z' as $X_1Y_1Z_1$ for convenience.





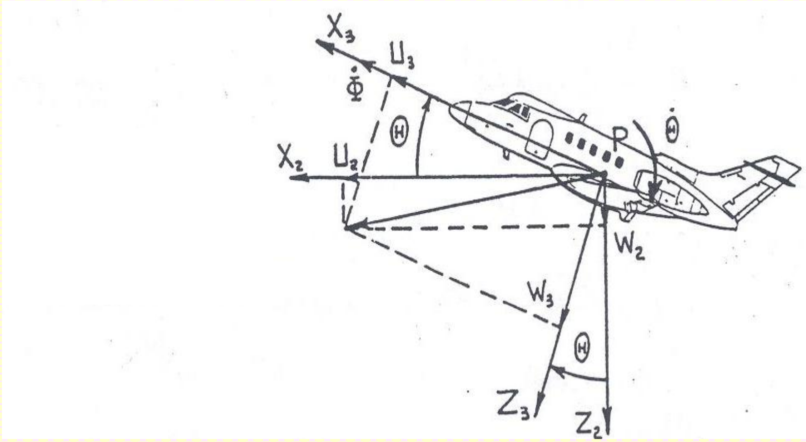
Rotate the system $X_1Y_1Z_1$ about Z_1 axis over an angle ψ

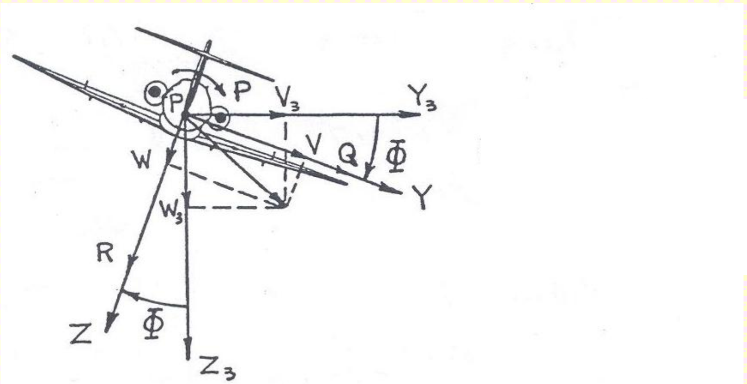
This yields the coordinate system $X_2Y_2Z_2$

Euler Angles

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

Rotate the system $X_2Y_2Z_2$ about Y_2 axis over an angle Θ
This yields the coordinate system $X_3Y_3Z_3$





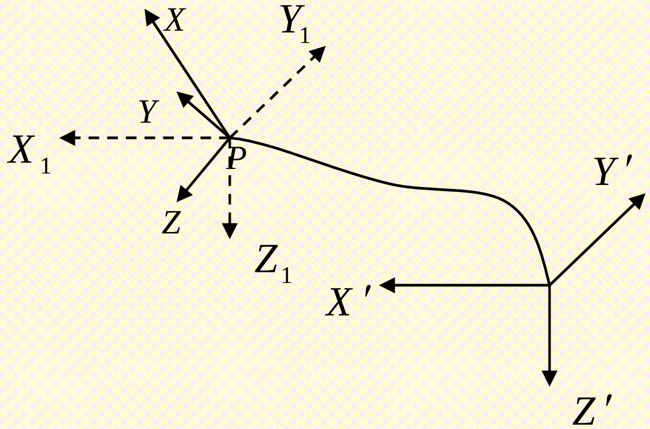
Rotate the system $X_3Y_3Z_3$ about Y_3 axis over an angle Φ
This yields the coordinate system XYZ

Flight Path Relative to Earth Fixed Coordinates (Inertial Frame)

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

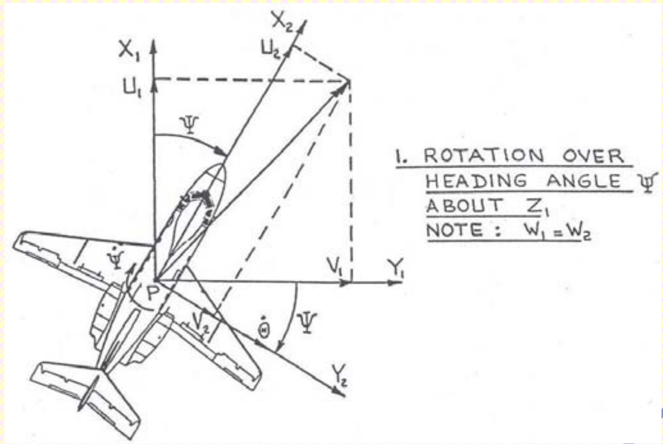
$$\underline{X'Y'Z' \quad \rightarrow \quad X_1 Y_1 Z_1}$$

$$\begin{bmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{bmatrix} = \begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix}$$



$$\underline{X_1 Y_1 Z_1 \quad \rightarrow \quad X_2 Y_2 Z_2}$$

$$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \\ W_2 \end{bmatrix}$$

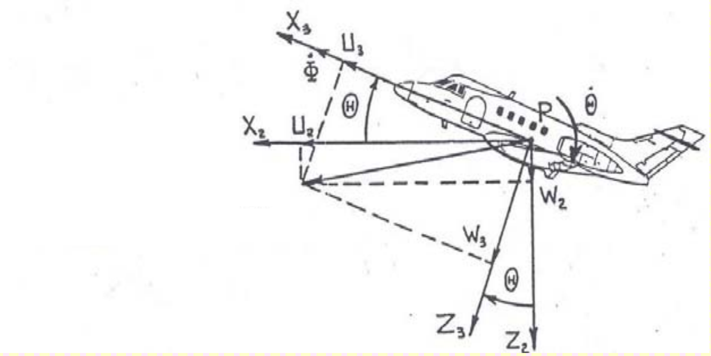


Flight Path Relative to Earth Fixed Coordinates (Inertial Frame)

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

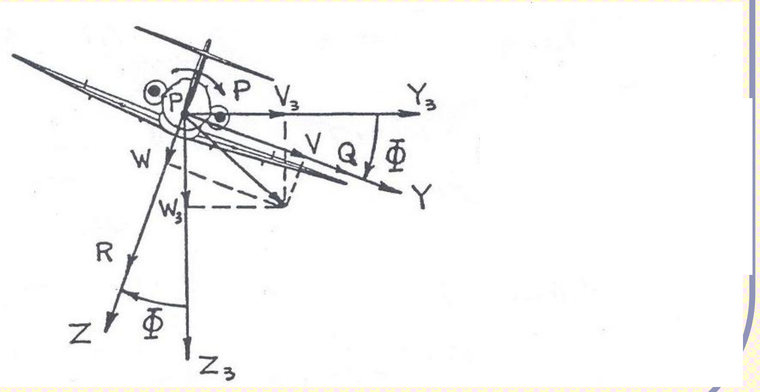
$$\underline{X_2 Y_2 Z_2 \quad \rightarrow \quad X_3 Y_3 Z_3}$$

$$\begin{bmatrix} U_2 \\ V_2 \\ W_2 \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} U_3 \\ V_3 \\ W_3 \end{bmatrix}$$



$$\underline{X_3 Y_3 Z_3 \quad \rightarrow \quad X\, Y\, Z}$$

$$\begin{bmatrix} U_3 \\ V_3 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$



Flight Path Relative to Earth Fixed Coordinates (Inertial Frame)

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{bmatrix} = \begin{bmatrix} \dot{x}' \\ y' \\ \dot{z}' \end{bmatrix} = \begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix}$$

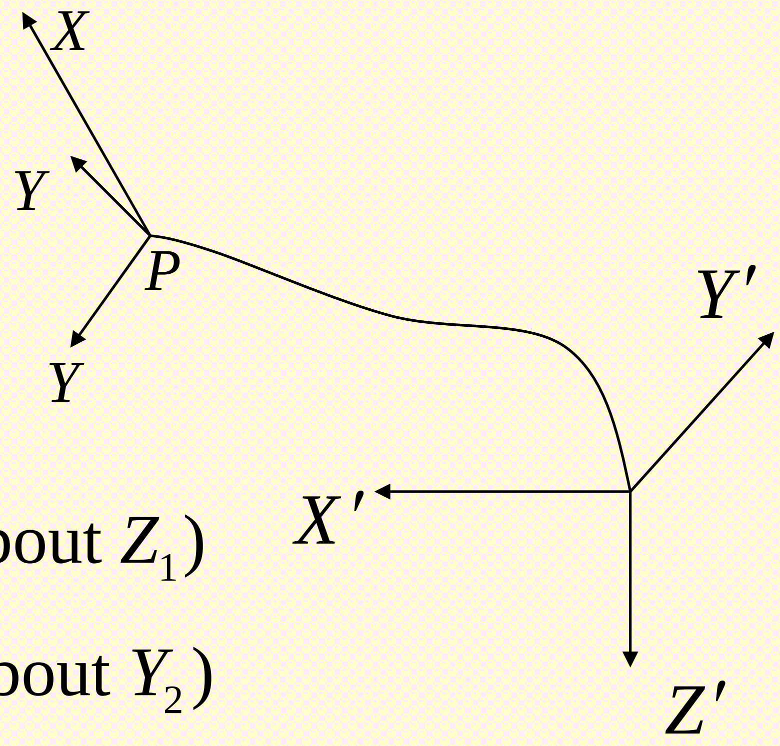
$$= \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_2 \\ V_2 \\ W_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} U_3 \\ V_3 \\ W_3 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

Relationship Between Ψ, Θ, Φ and P, Q, R

$$\vec{\omega} = iP + jQ + kR = \dot{\vec{\Psi}} + \dot{\vec{\Theta}} + \dot{\vec{\Phi}}$$



However,

$$\dot{\vec{\Psi}} = k_1\dot{\Psi} = k_2\dot{\Psi} \quad \text{(Rotation is about } Z_1\text{)}$$

$$\dot{\vec{\Theta}} = j_2\dot{\Theta} = j_3\dot{\Theta} \quad \text{(Rotation is about } Y_2\text{)}$$

$$\dot{\vec{\Phi}} = i_3\dot{\Phi} = i\dot{\Phi} \quad \text{(Rotation is about } X_3\text{)}$$

Relationship Between Ψ, Θ, Φ and P, Q, R

Using co-ordinate transformation rules, we can write:

$$k_2 = -i_3 \sin\Theta + k_3 \cos\Theta$$

$$\begin{bmatrix} j_3 \\ k_3 \end{bmatrix} = \begin{bmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix}$$

Using these relationships, we can write:

$$\vec{\omega} = \left\{ -i \sin\Theta + \cos\Theta \left(j \sin\Phi + k \cos\Phi \right) \right\} \dot{\Psi}$$

$$+ \left(j \cos\Phi - k \sin\Phi \right) \dot{\Theta} + i\dot{\Phi}$$

$$= iP + jQ + kR$$

Relationship Between Ψ, Θ, Φ and P, Q, R

Equating the coefficients,

$$P = \dot{\Phi} - \dot{\Psi}\sin\Theta$$

$$Q = \dot{\Theta}\cos\Phi + \dot{\Psi}\cos\Theta\sin\Phi$$

$$R = \dot{\Psi}\cos\Theta\cos\Phi - \dot{\Theta}\sin\Phi$$

In matrix form,

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \cos\Theta\sin\Phi \\ 0 & -\sin\Phi & \cos\Theta\cos\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}$$

Relationship Between Ψ, Θ, Φ and P, Q, R

Taking the inverse transformation,

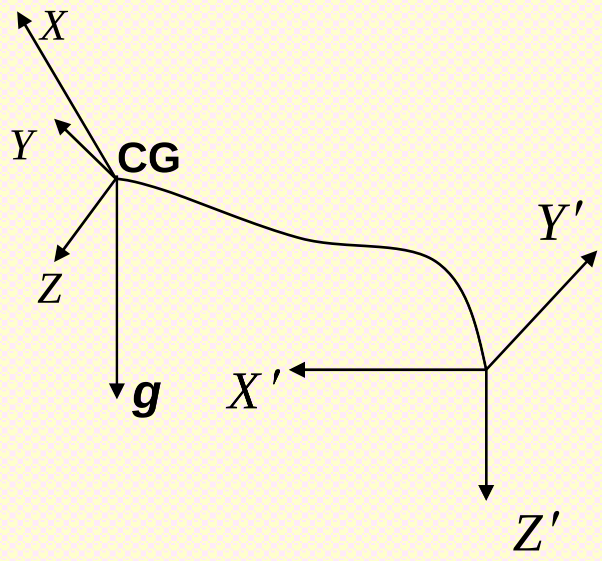
$$\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \cos\Theta\sin\Phi \\ 0 & -\sin\Phi & \cos\Theta\cos\Phi \end{bmatrix}^{-1} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta \\ Q\cos\Phi - R\sin\Phi \\ \left(Q\sin\Phi + R\cos\Phi\right)\sec\Theta \end{bmatrix}$$

Components of Gravitational Force in Body Co-ordinates

$$k'g = k_1 g = i g_X + j g_Y + k g_Z$$

It is desirable to express g_X, g_Y, g_Z in terms of g and Euler angles Θ and Φ.



Note that Ψ does not affect the component of gravity because $k_1 = k_2$.

Note that:

$$i_3 = i \qquad (i, j, k: \text{ Body frame})$$

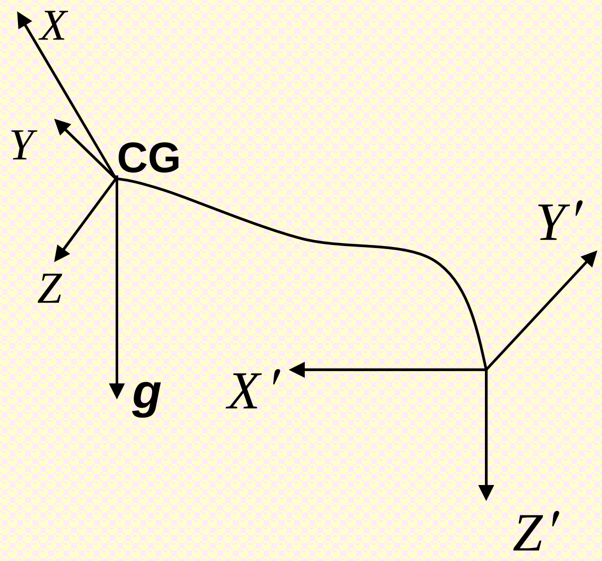
$$k_3 = k\cos\Phi + j\sin\Phi$$

Components of Gravitational Force

$$k_2 = -i_3 \sin\Theta + k_3 \cos\Theta \quad \text{(already derived)}$$
$$= -i_3 \sin\Theta + \left(k\cos\Phi + j\sin\Phi\right)\cos\Theta$$

But $k_1 = k_2$, we get:

$$k_1 g = g\left[-i\sin\Theta + \left(k\cos\Phi + j\sin\Phi\right)\cos\Theta\right]$$
$$= i g_X + j g_Y + k g_Z$$



Comparing the coefficients of i, j, k one can write:

$$g_X = -g\sin\Theta$$
$$g_Y = g\sin\Phi\cos\Theta$$
$$g_Z = g\cos\Phi\cos\Theta$$

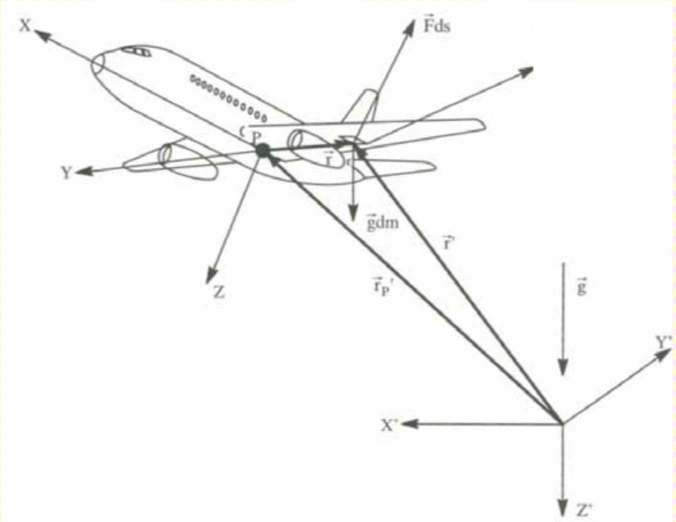
Kinematic Equations

Rate of Change of Euler Angles:

$$\dot{\Phi} = P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta$$

$$\dot{\Theta} = Q\cos\Phi - R\sin\Phi$$

$$\dot{\Psi} = \left(Q\sin\Phi + R\cos\Phi\right)\sec\Theta$$



Flight Path in the Inertial Frame:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{bmatrix} = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

Complete Six-DOF Model

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Airplane Dynamics: Six Degree-of-Freedom Model

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

$$\dot{U} = VR - WQ - g\sin\Theta + \frac{1}{m}(X + X_T)$$

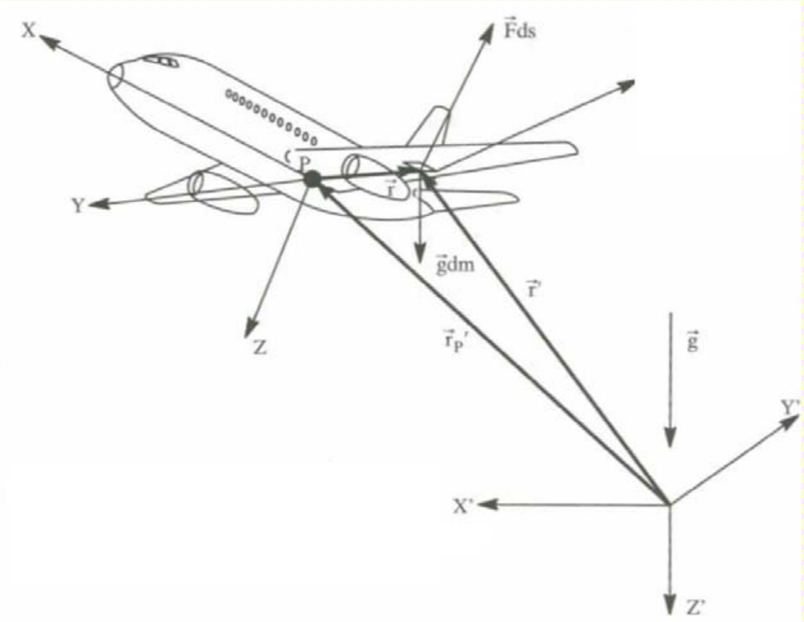
$$\dot{V} = WP - UR + g\sin\Phi\cos\Theta + \frac{1}{m}(Y + Y_T)$$

$$\dot{W} = UQ - VP + g\cos\Phi\cos\Theta + \frac{1}{m}(Z + Z_T)$$

$$\dot{P} = c_1 QR + c_2 PQ + c_3 (L + L_T) + c_4 (N + N_T)$$

$$\dot{Q} = c_5 PR - c_6 (P^2 - R^2) + c_7 (M + M_T)$$

$$\dot{R} = c_8 PQ - c_2 QR + c_4 (L + L_T) + c_9 (N + N_T)$$



$$\dot{\Phi} = P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta$$

$$\dot{\Theta} = Q\cos\Phi - R\sin\Phi$$

$$\dot{\Psi} = (Q\sin\Phi + R\cos\Phi)\sec\Theta$$

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$\dot{h} = -\dot{z}_I = U\sin\Theta - V\cos\Theta\sin\Phi - W\cos\Theta\cos\Phi$$

Airplane Dynamics: Six Degree-of-Freedom Model

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

where

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_8 \\ c_9 \end{bmatrix} = \frac{1}{\left(I_{xx}I_{zz} - I_{xz}{}^2\right)} \begin{bmatrix} I_{zz}(I_{yy} - I_{zz}) - I_{xz}{}^2 \\ I_{xz}(I_{zz} + I_{xx} - I_{yy}) \\ I_{zz} \\ I_{yz} \\ I_{xx}(I_{xx} - I_{yy}) + I_{xz}{}^2 \\ I_{xx} \end{bmatrix}$$

$$c_5 = I_{zz} - I_{xx}) / I_{yy}$$

$$c_6 = I_{xz} / I_{yy}$$

$$c_7 = 1 / I_{yy}$$

Airplane Dynamics: Six Degree-of-Freedom Model

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

$$X_T = \sum_{i=1}^{N} T_i \cos\Phi_{T_i} \cos\Psi_{T_i}$$

$$Y_T = \sum_{i=1}^{N} T_i \cos\Phi_{T_i} \sin\Psi_{T_i}$$

$$Z_T = -\sum_{i=1}^{N} T_i \sin\Phi_{T_i}$$

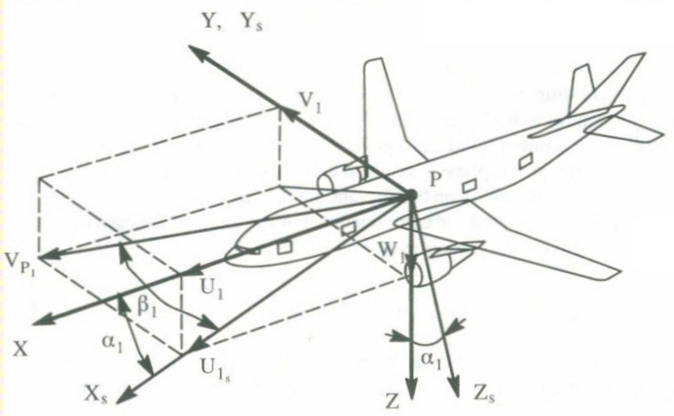
$$L_T = -\sum_{i=1}^{N}\left(T_i \cos\Phi_{T_i} \sin\Psi_{T_i}\right) z_{T_i} - \sum_{i=1}^{N}\left(T_i \sin\Phi_{T_i}\right) y_{T_i}$$

$$M_T = \sum_{i=1}^{N}\left(T_i \cos\Phi_{T_i} \cos\Psi_{T_i}\right) z_{T_i} + \sum_{i=1}^{N}\left(T_i \sin\Phi_{T_i}\right) x_{T_i}$$

$$N_T = -\sum_{i=1}^{N}\left(T_i \cos\Phi_{T_i} \cos\Psi_{T_i}\right) y_{T_i} + \sum_{i=1}^{N}\left(T_i \cos\Phi_{T_i} \sin\Psi_{T_i}\right) x_{T_i}$$

$$T_i = T_{i_{\max}} \cdot \sigma_{T_i}$$

$$\begin{bmatrix} X \\ Z \end{bmatrix} = T(\alpha)\begin{bmatrix} X_s \\ Z_s \end{bmatrix} = T(\alpha)(-\bar{q}S)\left(\begin{bmatrix} C_{D_O} & C_{D_\alpha} & C_{D_{i_h}} \\ C_{L_O} & C_{L_\alpha} & C_{L_{i_h}} \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \\ i_h \end{bmatrix} + \begin{bmatrix} C_{D_{\delta_E}} \\ C_{L_{\delta_E}} \end{bmatrix}\delta_E\right)$$



$$\begin{bmatrix} L \\ N \end{bmatrix} = T(\alpha)\begin{bmatrix} L_s \\ N_s \end{bmatrix} = T(\alpha)\,\bar{q}Sb\left(\begin{bmatrix} C_{l_\beta} \\ C_{n_\beta} \end{bmatrix}\beta + \begin{bmatrix} C_{l_{\delta_A}} & C_{l_{\delta_R}} \\ C_{n_{\delta_A}} & C_{n_{\delta_R}} \end{bmatrix}\begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix}\right)$$

$$Y = \bar{q}S\,C_Y = \bar{q}S\left(C_{Y_\beta}\beta + \begin{bmatrix} C_{Y_{\delta_A}} & C_{Y_{\delta_R}} \end{bmatrix}\begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix}\right)$$

$$M = \bar{q}S\bar{c}\,C_m = \bar{q}S\bar{c}\begin{bmatrix} C_{m_o} & C_{m_\alpha} & C_{m_{i_h}} \end{bmatrix}\begin{bmatrix} 1 & \alpha & i_h \end{bmatrix}^T + C_{m_{\delta_E}}\delta_E$$

$$T(\alpha) \triangleq \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Six Degree-of-Freedom Model: Important Observations

- The Six-DOF model consists of 12 equations, out of which 3 equations (namely x, y and ψ) are decoupled with the non-rotating and flat earth assumptions. Hence, for flight control design, usually 9 equations are good enough.
- One still need to integrate the x, y and ψ equations, however, to know the trajectory of the vehicle in the inertial frame.
- The 9 equations are further reduced to 8 equations (assuming constant height and neglecting the z equation) and then decoupled into 4 longitudinal equations and 4 lateral equations in the linearized (small disturbance) theory.

Six Degree-of-Freedom Model: Important Observations

- An airplane is symmetric about its XZ-plane. Hence:

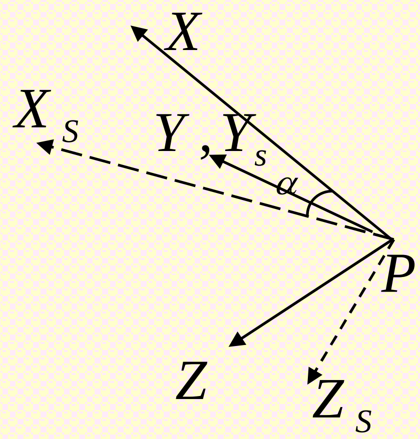
$$I_{xy} = I_{yz} = 0$$

- Missiles and launch vehicles are typically symmetric about both XZ-plane as well as XY-plane. Hence:

$$I_{xy} = I_{yz} = I_{zx} = 0$$

Other Reference Frames used in Flight Dynamics

- Stability Frame
  - Body frame is rotated by α about the Y-axis



- Wind Frame
  - Stability frame is further rotated by β about stability Z_s-axis
- Note: Body, Stability and Wind frame variables are related through rotational transformations.

Small-Disturbance Flight Dynamics

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

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Linearization using Small Perturbation Theory

Perturbation in the variables:

$$U = U_0 + \Delta U \qquad V = V_0 + \Delta V \qquad W = W_0 + \Delta W$$

$$P = P_0 + \Delta P \qquad Q = Q_0 + \Delta Q \qquad R = R_0 + \Delta R$$

$$X = X_0 + \Delta X \qquad Y = Y_0 + \Delta Y \qquad Z = Z_0 + \Delta Z$$

$$X_T = X_{T_0} + \Delta X_T \quad Y_T = Y_{T_0} + \Delta Y_T \qquad Z_T = Z_{T_0} + \Delta Z$$

$$M = M_0 + \Delta M \quad N = N_0 + \Delta N \qquad L = L_0 + \Delta L$$

$$\Phi = \Phi_0 + \Delta \phi \qquad \Theta = \Theta_0 + \Delta \theta \qquad \Psi = \Psi_0 + \Delta \psi$$

$$\delta_A = \delta_{A_0} + \Delta \delta_A \quad \delta_E = \delta_{E_0} + \Delta \delta_E \quad \delta_R = \delta_{R_0} + \Delta \delta_R$$

Trim Condition for Straight and Level Flight

- Assume: $V_0 = P_0 = Q_0 = R_0 = \Phi_0 = \underbrace{Y_{T_0} = Z_{T_0} = 0}_{\text{Typically True } \forall t}$
- Select: $X_{T_0},\ z_{I_0}\ \ (i.e.\ h_0)$
- Enforce: $\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = \dot{\Phi} = \dot{\Theta} = \dot{z}_I = 0$
- Solve for: $U_0, W_0, X_0, Y_0, Z_0, L_0, M_0, N_0, \Theta_0$
- Verify: $Y_0 = L_0 = M_0 = N_0 = 0$

Linearization using Small Perturbation Theory

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

$$\Delta X = \frac{\partial X}{\partial U}\Delta U + \frac{\partial X}{\partial W}\Delta W + \frac{\partial X}{\partial \delta_E}\Delta \delta_E + \frac{\partial X}{\partial \delta_T}\Delta \delta_T$$

$$\Delta Y = \frac{\partial Y}{\partial V}\Delta V + \frac{\partial Y}{\partial P}\Delta P + \frac{\partial Y}{\partial R}\Delta R + \frac{\partial Y}{\partial \delta_R}\Delta \delta_R$$

$$\Delta Z = \frac{\partial Z}{\partial U}\Delta U + \frac{\partial Z}{\partial W}\Delta W + \frac{\partial Z}{\partial \dot{W}}\Delta \dot{W} + \frac{\partial Z}{\partial Q}\Delta Q + \frac{\partial Z}{\partial \delta_E}\Delta \delta_E + \frac{\partial Z}{\partial \delta_T}\Delta \delta_T$$

$$\Delta L = \frac{\partial L}{\partial V}\Delta V + \frac{\partial L}{\partial P}\Delta P + \frac{\partial L}{\partial R}\Delta R + \frac{\partial L}{\partial \delta_R}\Delta \delta_R + \frac{\partial L}{\partial \delta_A}\Delta \delta_A$$

$$\Delta M = \frac{\partial M}{\partial U}\Delta U + \frac{\partial M}{\partial W}\Delta W + \frac{\partial M}{\partial \dot{W}}\Delta \dot{W} + \frac{\partial M}{\partial Q}\Delta Q + \frac{\partial M}{\partial \delta_E}\Delta \delta_E + \frac{\partial M}{\partial \delta_T}\Delta \delta_T$$

$$\Delta N = \frac{\partial N}{\partial V}\Delta V + \frac{\partial N}{\partial P}\Delta P + \frac{\partial N}{\partial R}\Delta R + \frac{\partial N}{\partial \delta_R}\Delta \delta_R + \frac{\partial N}{\partial \delta_A}\Delta \delta_A$$

State Variable Representation of Longitudinal Dynamics

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

State space form:

$$\dot{X} = AX + BU_c$$

$$A = \begin{bmatrix} X_U & X_W & 0 & -g \\ Z_U & Z_W & U_0 & 0 \\ M_U + M_{\dot{W}} Z_U & M_W + M_{\dot{W}} Z_W & M_Q + M_{\dot{W}} U_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} \Delta U \\ \Delta W \\ \Delta Q \\ \Delta\theta \end{bmatrix}$$

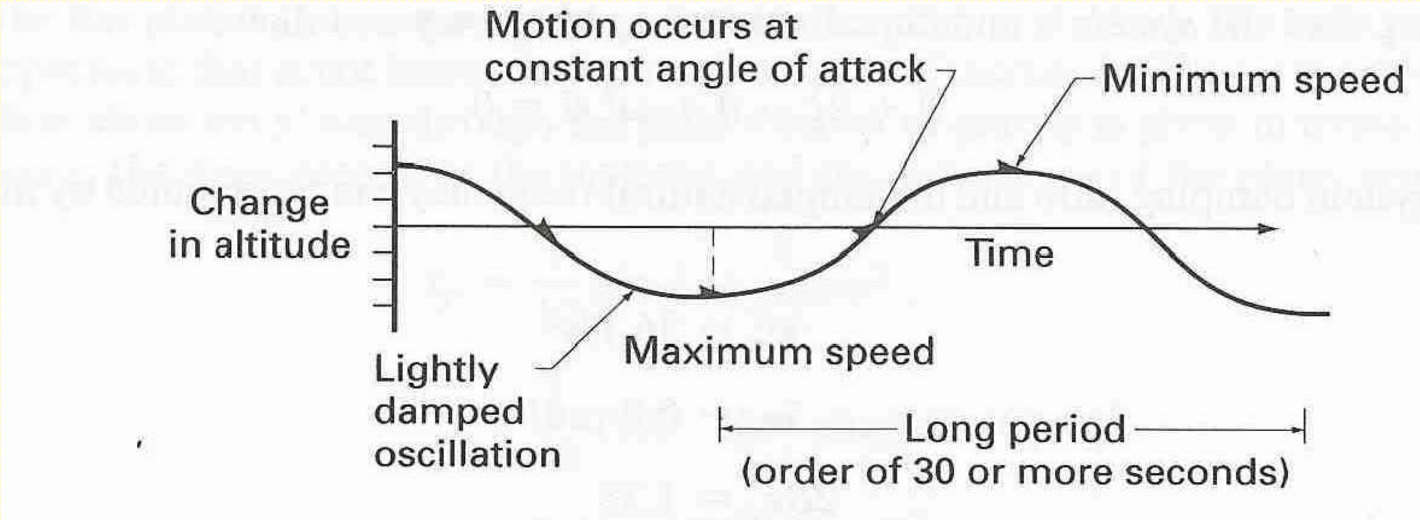
$$B = \begin{bmatrix} X_{\delta_E} & X_{\delta_T} \\ Z_{\delta_E} & Z_{\delta_T} \\ M_{\delta_E} + M_{\dot{W}} Z_{\delta_E} & M_{\delta T} + M_{\dot{W}} Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \qquad U_c = \begin{bmatrix} \Delta\delta_E \\ \Delta\delta_T \end{bmatrix}$$

$$X_U = \frac{1}{m}\left(\frac{\partial X}{\partial U}\right), \quad X_W = \frac{1}{m}\left(\frac{\partial X}{\partial W}\right) \quad etc.$$

Phugoid Mode

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

- Lightly damped
- Changes in pitch attitude, altitude, velocity
- Constant angle of attack



Phugoid Mode

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

$$\Delta\alpha = \frac{\Delta W}{U_0}$$

$$\Delta\alpha = 0 \quad \Rightarrow \quad \Delta W = 0$$

State Equations: $$\begin{bmatrix} \Delta\dot{U} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_U & -g \\ \frac{Z_U}{U_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta U \\ \Delta\theta \end{bmatrix}$$

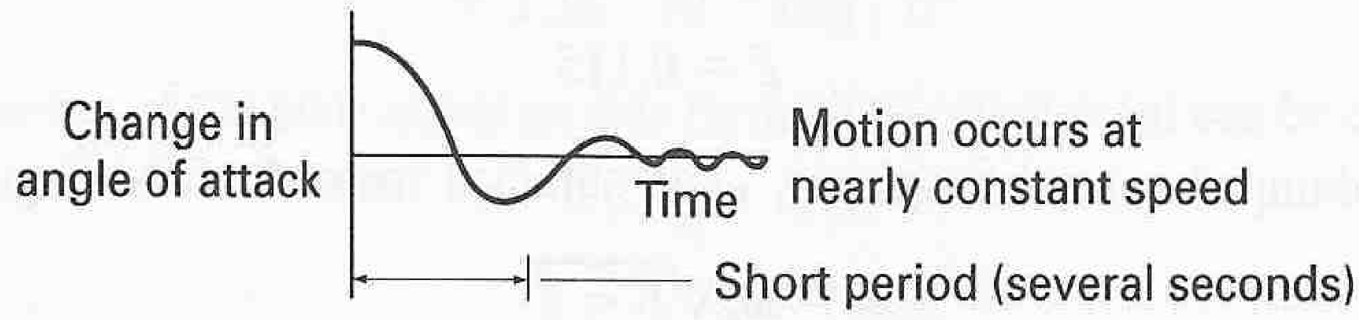
Frequency: $$\omega_{n_p} = \sqrt{\frac{-Z_U\, g}{U_0}}$$

Damping ratio: $$\zeta_P = \frac{-X_U}{2\omega_{n_P}}$$

Short Period Mode

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

- Heavily damped
- Short time period
- Constant velocity



Short Period Mode

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

$$\Delta U = 0$$

State Equations: $\begin{bmatrix} \Delta\dot{W} \\ \Delta\dot{Q} \end{bmatrix} = \begin{bmatrix} Z_W & U_0 \\ M_W + M_{\dot{W}} Z_W & M_Q + M_{\dot{W}} U_0 \end{bmatrix} \begin{bmatrix} \Delta W \\ \Delta Q \end{bmatrix}$

Frequency: $\omega_{n_{SP}} = \sqrt{\left(\dfrac{Z_\alpha M_Q}{U_0} - M_\alpha \right)}$

Damping Ratio: $\zeta_{SP} = \dfrac{M_Q + M_{\dot{\alpha}} + \dfrac{Z_\alpha}{U_0}}{2\omega_{n_{SP}}}$

State Variable Representation of Lateral Dynamics

State space form: $\dot{X} = AX + BU_c$

$$A = \begin{bmatrix} Y_V & Y_P & -(U_0 - Y_R) & g\cos\theta_0 \\ L_V{}^* + \dfrac{I_{XZ}}{I_X} N_V{}^* & L_P{}^* + \dfrac{I_{XZ}}{I_X} N_P{}^* & L_R{}^* + \dfrac{I_{XZ}}{I_X} N_R{}^* & 0 \\ N_V{}^* + \dfrac{I_{XZ}}{I_Z} L_V{}^* & N_P{}^* + \dfrac{I_{XZ}}{I_Z} L_P{}^* & N_R{}^* + \dfrac{I_{XZ}}{I_Z} L_R{}^* & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} \Delta V \\ \Delta P \\ \Delta R \\ \Delta\phi \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & Y_{\delta_R} \\ L_{\delta_A}{}^* + \dfrac{I_{XZ}}{I_X} N_{\delta_A} & L_{\delta_R}{}^* + \dfrac{I_{XZ}}{I_X} N_{\delta_R} \\ N_{\delta_A}{}^* + \dfrac{I_{XZ}}{I_Z} L_{\delta_A}{}^* & N_{\delta_R}{}^* + \dfrac{I_{XZ}}{I_Z} L_{\delta_R}{}^* \\ 0 & 0 \end{bmatrix} \qquad U_c = \begin{bmatrix} \Delta\delta_A \\ \Delta\delta_R \end{bmatrix}$$

State Variable Representation of Lateral-directional Dynamics

Note: If $I_{XZ} = 0$, then

$$A = \begin{bmatrix} Y_V & Y_P & -(u_0 - Y_R) & g\cos\theta_0 \\ L_V & L_P & L_R & 0 \\ N_V & N_P & N_R & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & Y_{\delta_R} \\ L_{\delta_A} & L_{\delta_R} \\ N_{\delta_A} & N_{\delta_R} \\ 0 & 0 \end{bmatrix}$$

Aircraft Responses

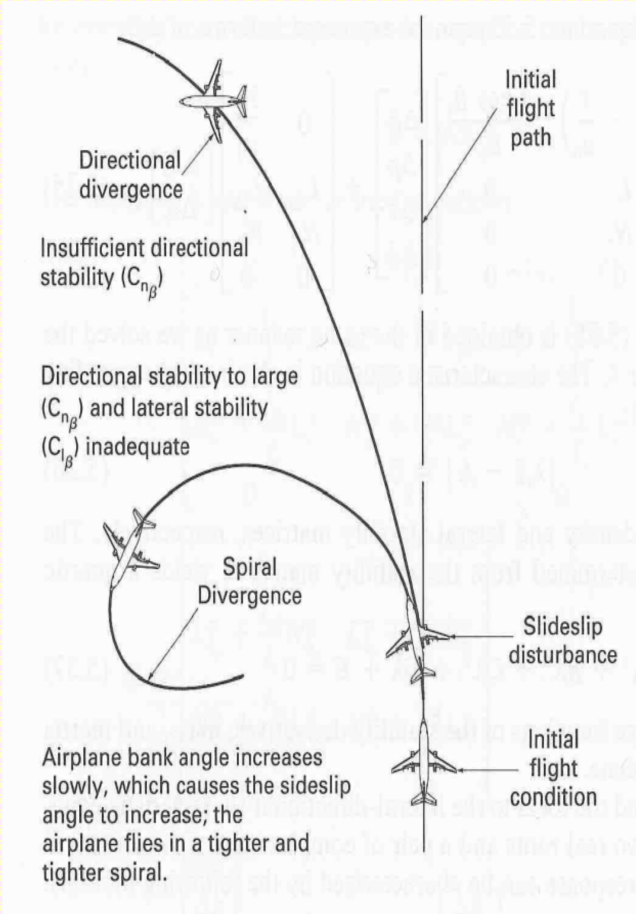
- Spiral Mode: Slowly convergent or divergent motion
- Rolling Mode: Highly convergent motion
- Dutch Roll Mode: Lightly damped oscillatory motion having low frequency

Lateral Dynamic Instabilities

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

Directional divergence

- Do not possess directional stability
- Tend towards ever-increasing angle of sideslip
- Largely controlled by rudder



Spiral divergence

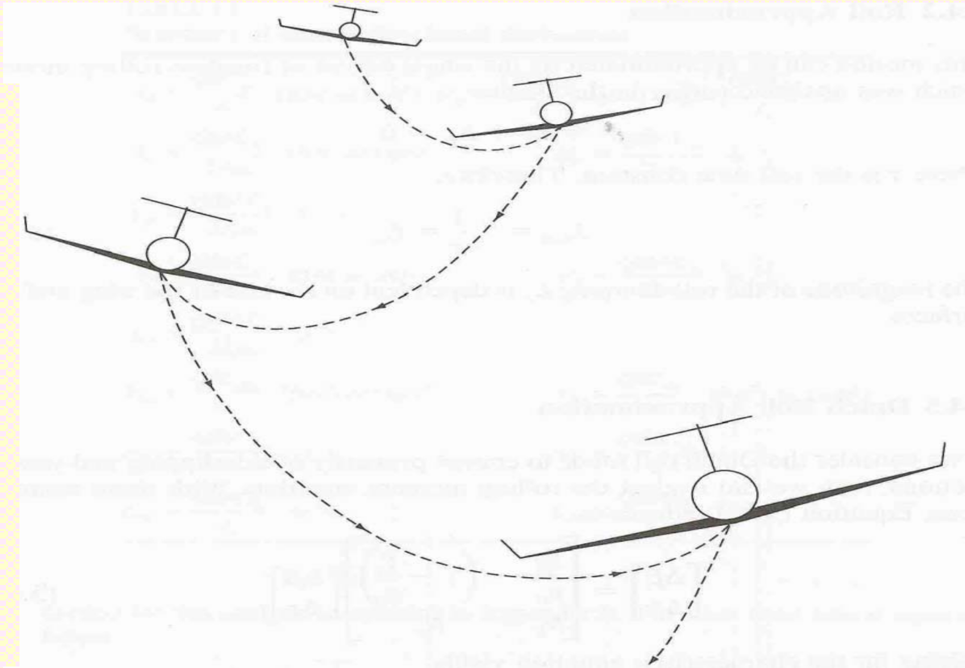
- Spiral divergence tends to gradual spiraling motion & leads to high speed spiral dive
- Non–oscillatory divergent motion
- Largely controlled by ailrons

Lateral Dynamic Instabilities

Reference: R. C. Nelson, Flight Stability and Automatic Control, McGraw-Hill, 1989.

Dutch roll oscillation

- Coupled directional-spiral oscillation
- Combination of rolling and yawing oscillation of same frequency but out of phase each other
- Time period can be of 3 to 15 sec



- Yaw damper is used for improving the system damping and used to improves both spiral and dutch roll characteristics

Attitude Representation

Reference: H. Schaub and J. L. Junkins, Analytical Mechanics of Space Systems, AIAA, 2003.

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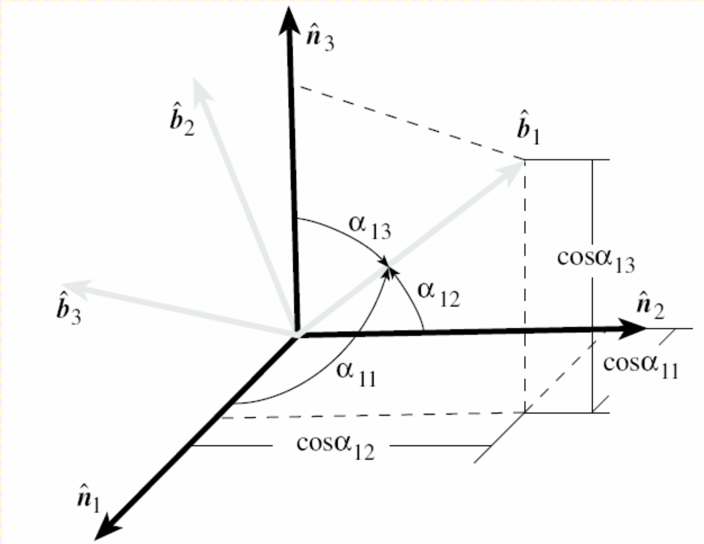
Attitude Representation

- **Definition:** Attitude coordinates are set of parameters that completely describes the orientation of a rigid body relative to some reference frame.
- Various Possibilities:
  - Euler Angles
  - Direction Cosines
  - Quaternions (Euler parameters)
- Broad Class:
  - Three parameter representation
  - Four parameter representation (quaternions)

Direction cosine matrix

Let the two refrence frames N and B each be defined through sets of orthonormal right-handed sets of vectors $\{\hat{\mathrm{n}}\}$ and $\{\hat{\mathrm{b}}\}$ where we use the shorthand vectrix notation



$$\{\hat{\mathrm{b}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{n}\}$$

$\{\hat{\mathrm{b}}\} = [C]\{\hat{n}\}$, C is called the "direction cosine matrix", $C_{ij} = \cos(\angle \hat{b}_i, \hat{n}_j) = \hat{b}_i \bullet \hat{n}_j$

Properties of DCM

- Direction cosine matrix [C] is orthogonal, $[C][C]^T = [I_{3x3}]$

$$\left\{\hat{n}\right\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{21} & \cos\alpha_{31} \\ \cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32} \\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33} \end{bmatrix} \left\{\hat{b}\right\} = [C]^T \left\{\hat{b}\right\}, \left\{\hat{b}\right\} = [C][C]^T \left\{\hat{b}\right\}$$

- Inverse of [C] is the transpose of [C], $[C]^{-1}=[C]^T$
- Determinant of DCM is ± 1, $det(CC^T)= det([I_{3x3}])=1$
- Direction cosine matrix is the most fundamental, but highly redundant, method of describing a relative orientation.
- Minimum 3 parameters are used to describe a reference frame orientation, it has 9 entries, hence 6 extra parameters are redundant through orthogonality condition.

Euler angles

- Standard set of Euler angles used in aircraft and missile is (3-2-1)i.e. (ψ,θ,Φ). This sequence is called asymmetric set.
- Other is (3-1-3) set of Euler angles, to define the orientation of orbit planes of the planets relative to the earth orbit's plane. It is called symmetric set.
- DCM can be parameterized in terms of the Euler angles.
- Each Euler angle defines a successive rotation about one of the body axes

Euler angles

- The direction cosine matrix in terms of the (3-2-1)Euler angles is

$$C = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}, \text{ where } c\theta = \cos\theta,\ s\theta = \sin\theta$$

$$\psi = \theta_1 = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right), \quad \theta = \theta_1 = -\sin^{-1}\left(C_{13}\right), \quad \phi = \theta_3 = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right)$$

- Euler angles provide a compact,3 parameter attitude description whose coordinates are easy to visualize

Quaternion

- Another popular set of attitude coordinates are the four Euler parameters (quaternions-*4D* vector space).
- They provide a redundant, **non-singular** attitude description and are well suited to describe arbitrary, large rotations.

$$q = q_0 + \vec{q}, \qquad k = ij = -ji,\ i = jk = -kj,\ j = ki = -ik$$

$$= q_0 + iq_1 + jq_2 + kq_3 \qquad i^2 = j^2 = k^2 = ijk = -1$$

Equality: $p = q \Leftrightarrow p_0 = q_0,\ p_1 = q_1,\ p_2 = q_2,\ p_3 = q_3$

Addition: $p+q=\left(p_0+q_0\right)+i\left(p_1+q_1\right)+j\left(p_2+q_2\right)+k\left(p_3+q_3\right)$

Multiplication: $cq = cq_0 + cq_1 i + cq_2 j + cq_3 k$

Quaternion

Conjugate of quaternion : $q^* = q_0 - iq_1 - jq_2 - kq_3$

Norm of quaternion : $|q| = \sqrt{q^* q}, \quad |q|^2 = q^* q, \quad |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$

Unit quaternion (normalized quaternion) : A unit quaternion, *q*, is a quaternion such that |*q*| = 1.

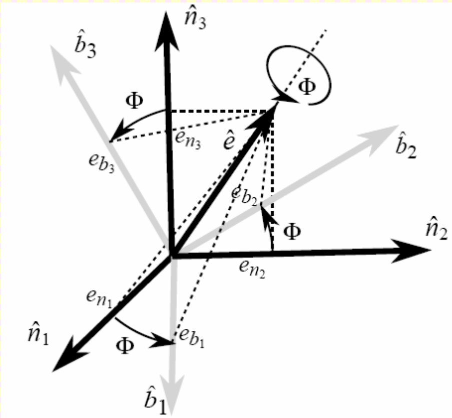
Inverse of quaternion: $q^{-1} q = q q^{-1} = 1, \quad q^{-1} q q^* = q^* q q^{-1} = q^*$

$$q^{-1} = \frac{q^*}{|q|^2}$$

Quaternion: Principal rotation vector

Theorem:

A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the "principal axis".



The Euler parameter vector is defined in terms of the principal rotation elements as

$$q_0 = \cos\left(\Phi/2\right),\ q_1 = e_1 \sin\left(\Phi/2\right), q_2 = e_2 \sin\left(\Phi/2\right),\ q_3 = e_3 \sin\left(\Phi/2\right)$$

Quaternion

- The constraint equation in quaternion algebra (a holonomic constraint) geometrically describes a four-dimensional unit sphere. Any rotation described through the Euler parameters has a trajectory on the surface of this constraint sphere.
- Euler angles to Quaternion:

$$q_0 = \cos\left(\psi/2\right)\cos\left(\theta/2\right)\cos\left(\Phi/2\right) + \sin\left(\psi/2\right)\sin\left(\theta/2\right)\sin\left(\Phi/2\right)$$

$$q_1 = \cos\left(\psi/2\right)\cos\left(\theta/2\right)\sin\left(\Phi/2\right) - \sin\left(\psi/2\right)\sin\left(\theta/2\right)\cos\left(\Phi/2\right)$$

$$q_2 = \cos\left(\psi/2\right)\sin\left(\theta/2\right)\cos\left(\Phi/2\right) + \sin\left(\psi/2\right)\cos\left(\theta/2\right)\sin\left(\Phi/2\right)$$

$$q_3 = \sin\left(\psi/2\right)\cos\left(\theta/2\right)\cos\left(\Phi/2\right) - \cos\left(\psi/2\right)\sin\left(\theta/2\right)\sin\left(\Phi/2\right)$$

- Quaternions to DCM:

$$[C] = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Thanks for the Attention….!