Lecture – 4

Classical Control Overview – II

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Stability Analysis through Transfer Function

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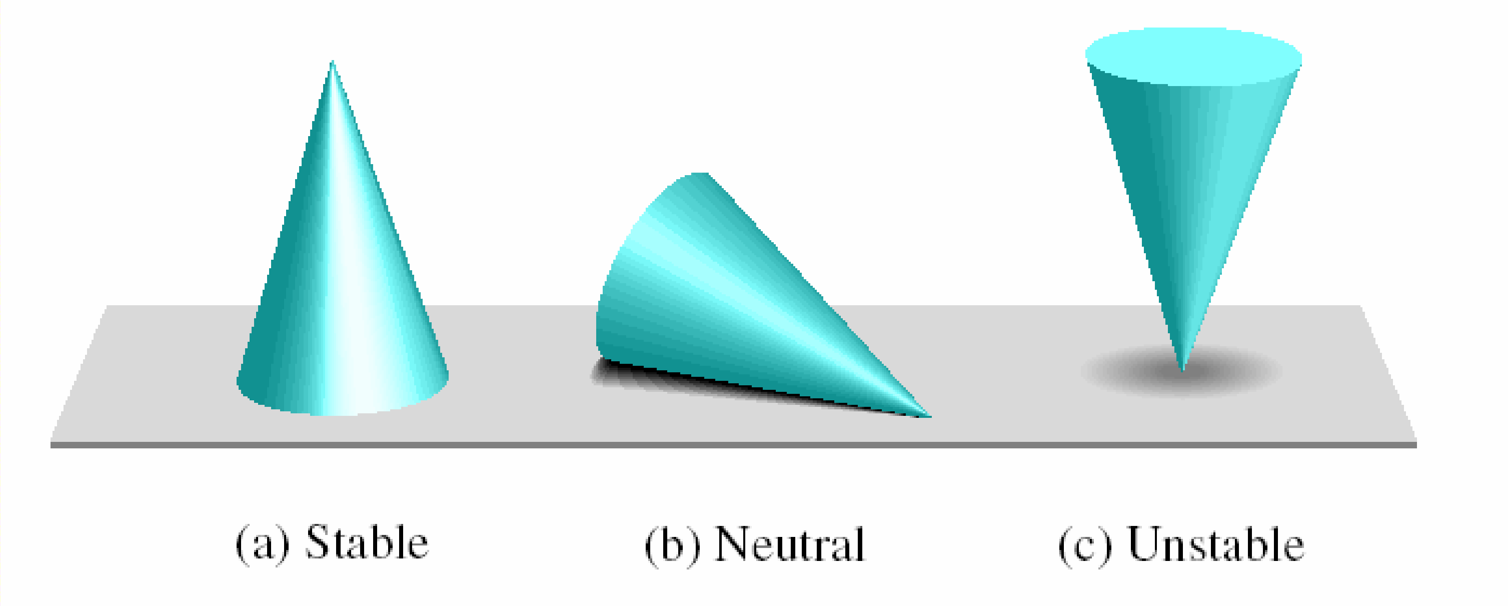
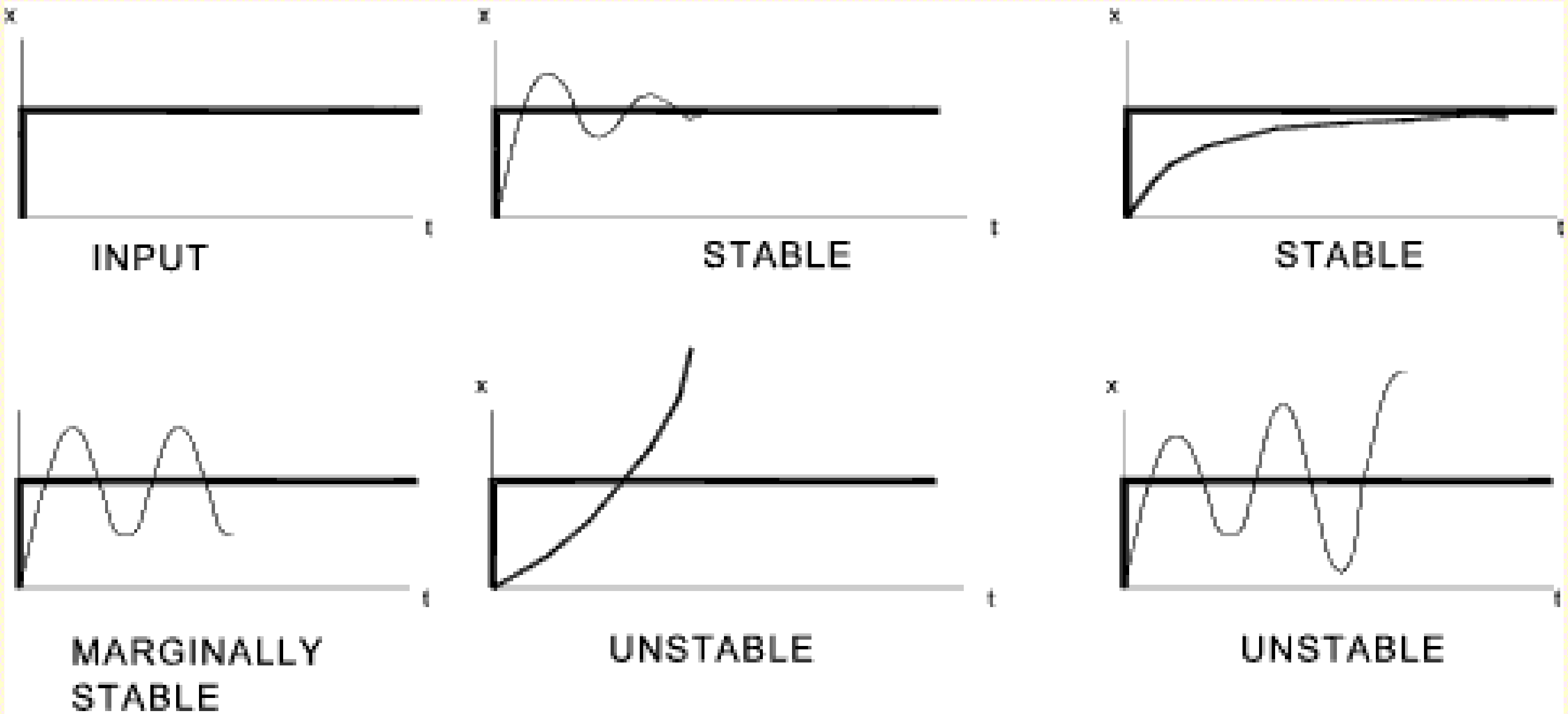
Introduction

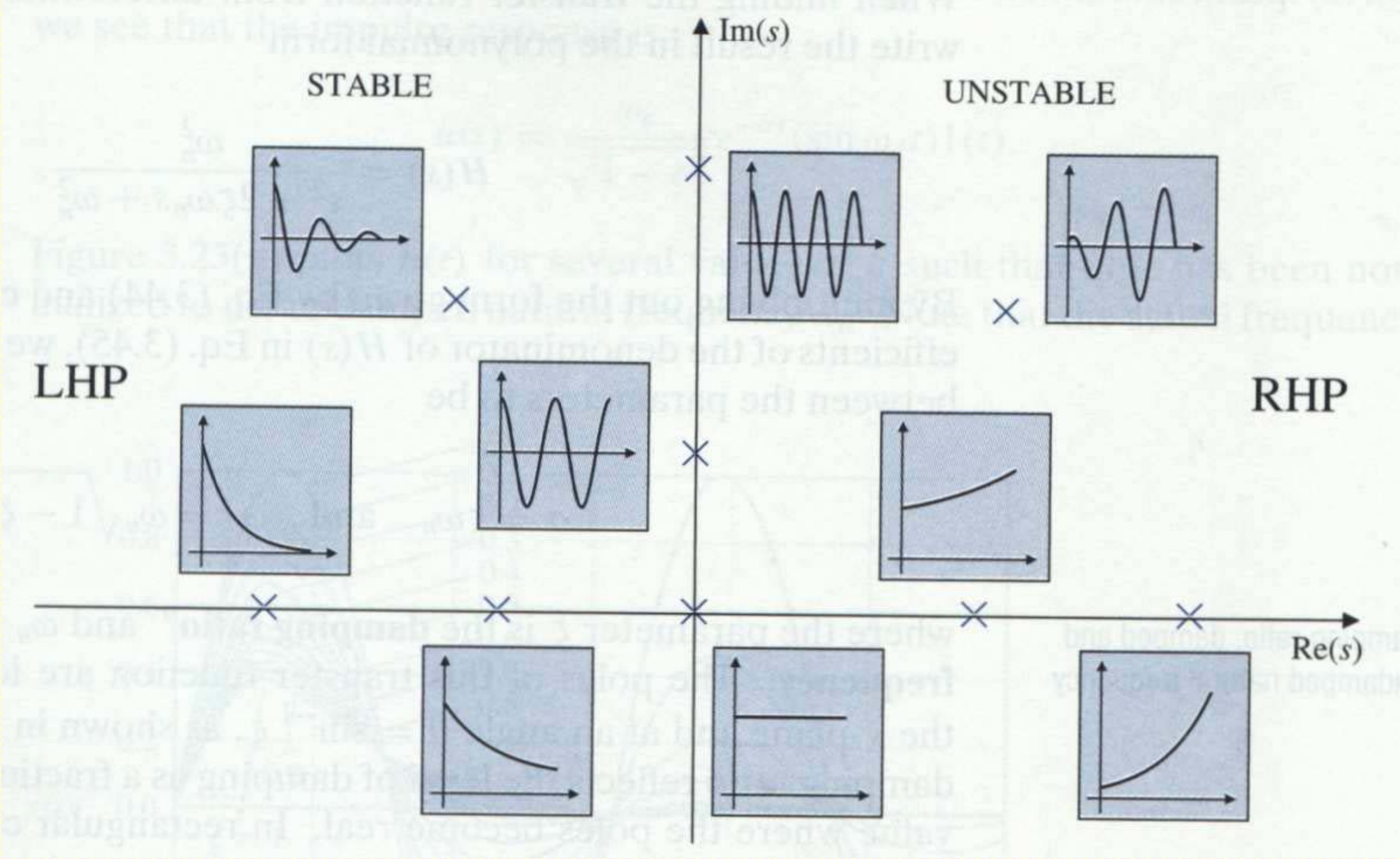


Conceptual description of linear system stability

Introduction



Introduction



Introduction

Total response of a system is the sum of transient and steady state responses; i.e.

$$c(t) = c_{transient}(t) + c_{staedystate}(t)$$

$c_{transient}(t)$ is the response that goes from a initial state to the final state as t evolves.

$c_{staedystate}(t)$ is the response as $t \to \infty$

Stability Definition

- A system is **Stable** if the natural response approaches zero as time approaches infinity .
- A system is **Unstable** if the natural response approaches infinity as time approaches infinity.
- A system is **Marginally Stable** if the natural response neither decays nor grows but remains constant or oscillates within a bound

Definition (BIBO Stability)

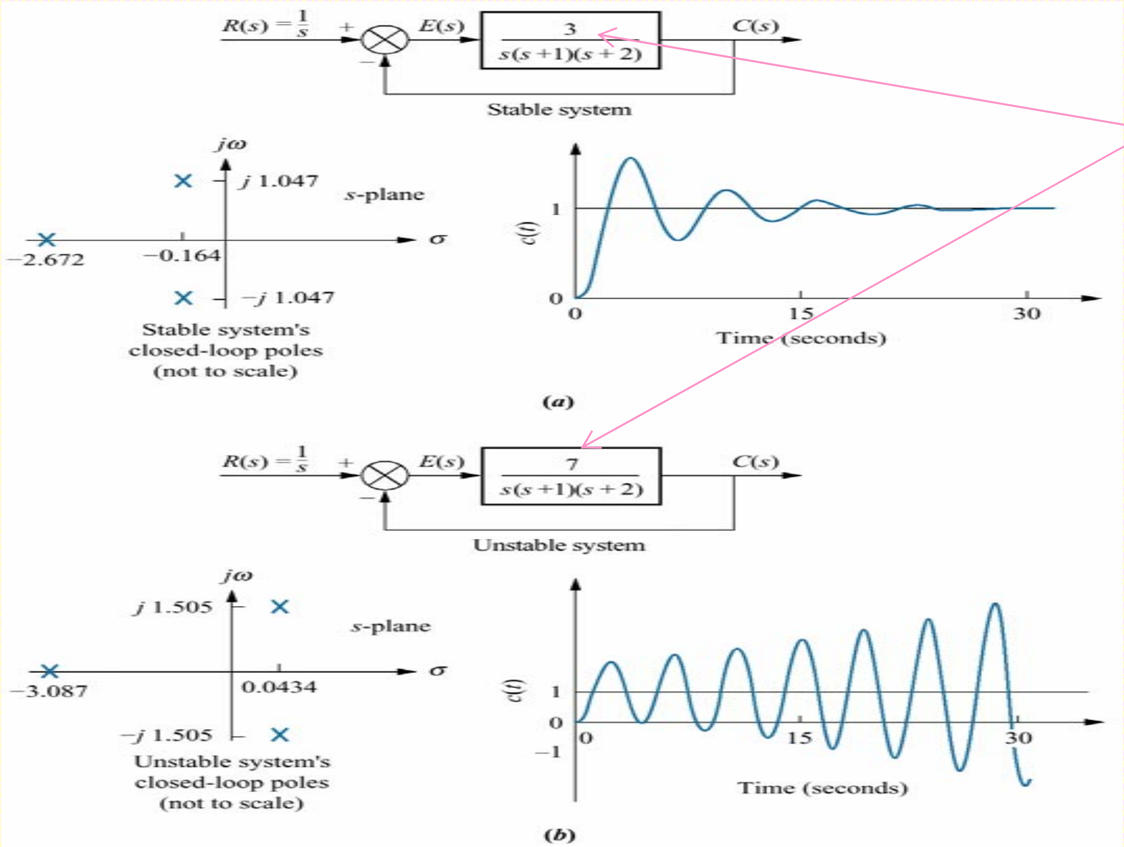
- A system is **Stable** if every bounded input yields a bounded output.
- A system is **Unstable** if any bounded input yields an unbounded output.

Note: For linear systems, both notions of stability are equivalent.

Stability Analysis from Closed Loop Transfer function

- **Stable systems** have closed-loop transfer functions with poles only in the left half-plane.
- **Unstable systems** have closed-loop transfer functions with at least one pole in the right half plane and/or poles of multiplicity greater than one on the imaginary axis.
- **Marginally Stable** systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.

Stability of closed loop system with gain variation



As gain is increased from 3 to 7, the system becomes unstable

Routh–Hurwitz Approach for Stability Analysis

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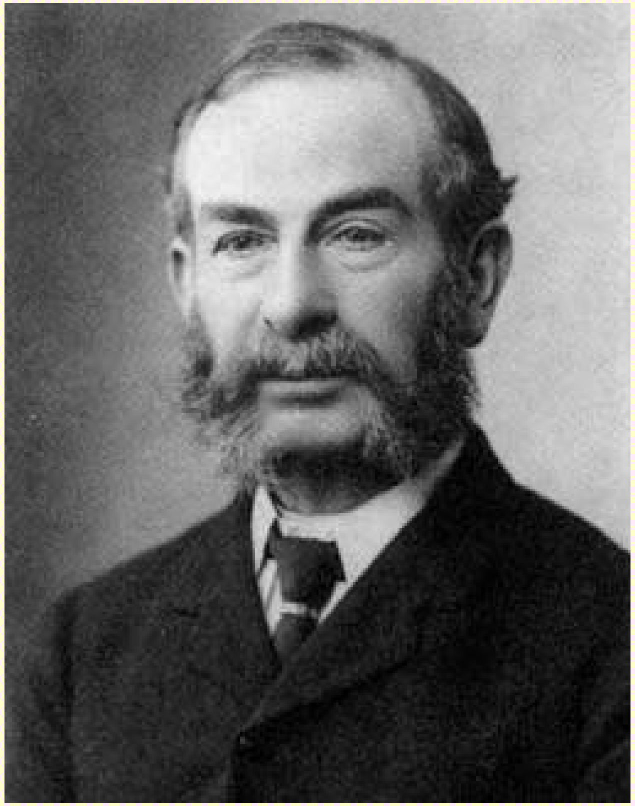
Sufficient Conditions for Instability and Marginal Stability:

- A system is **<u>Unstable</u>** if all signs of the coefficients of the denominator of the closed loop transfer function are ***not same***.
- If powers of *s* are missing from the denominator of the closed loop transfer function, then the system is either **<u>Unstable</u>** or at best **<u>Marginally Stable</u>**

<u>Question:</u> What if all coefficients are positive and no power of *s* is missing?

<u>Answer:</u> Routh-Hurwitz criterion.

Routh – Hurwitz



Edward Routh, 1831 (Quebec)-1907 (Cambridge, England)

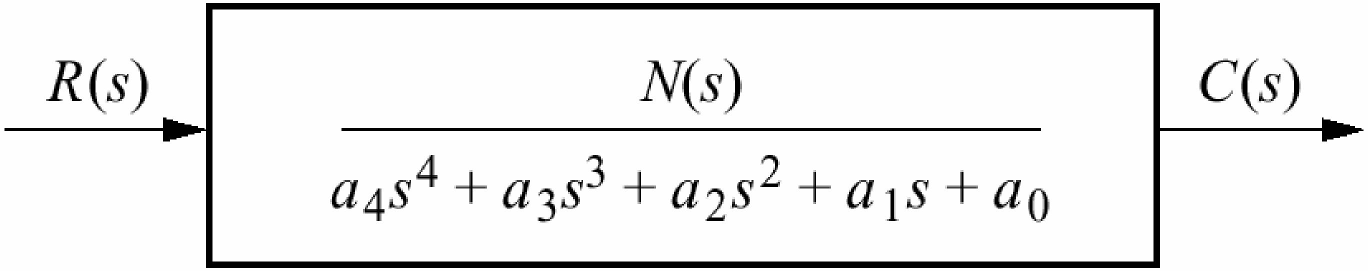
Adolf Hurwitz, 1859 (Germany)-1919 (Zurich)

Routh–Hurwitz Criterion

- **Caution:** This method tells how many closed loop system poles are in the left-half plane, in the right-half plane and on the *jω* axis. However, it does not tell the location of the poles.

- Methodology
  - Construct a Routh table
  - Interpret the Routh table: The sign of the entries of the first column imbeds the information about the stability of the closed loop system.

Generating Routh Table from Closed loop Transfer Function



$$\frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Initial layout for Routh table:

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Generating Routh Table :

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Completed Routh table

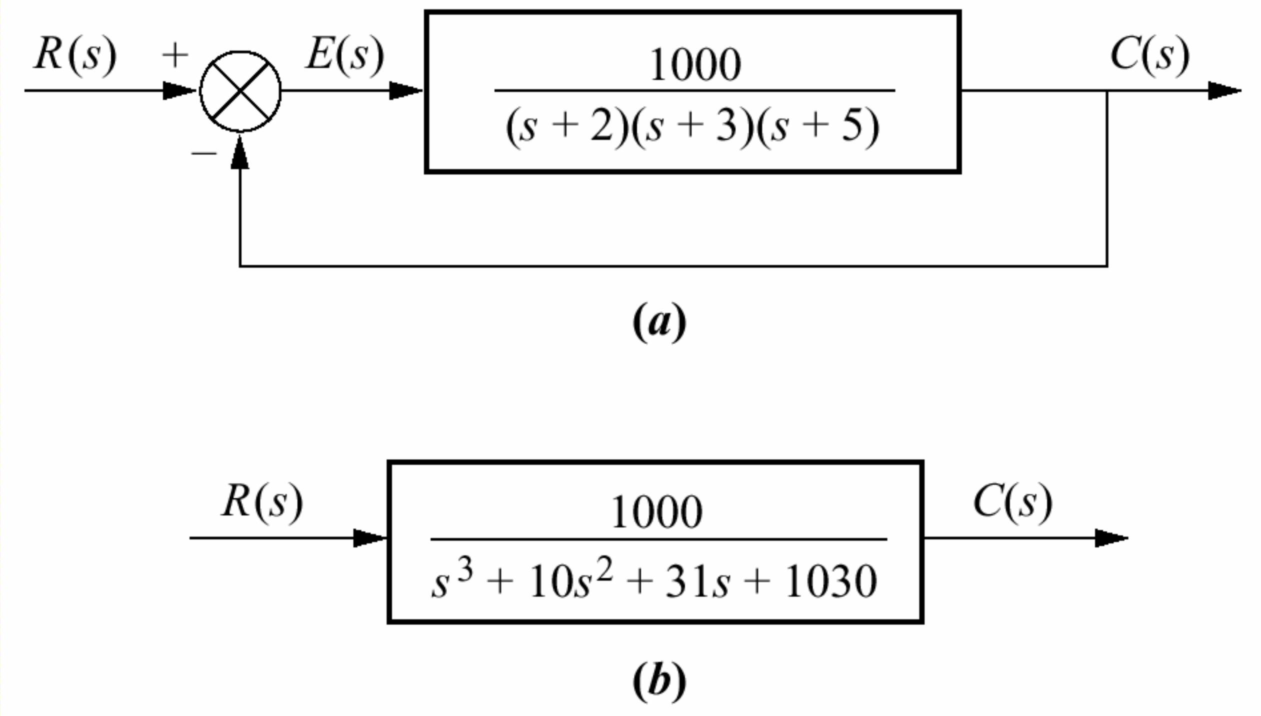
Note: Any row of the Routh table can be multiplied by a “positive” constant

Interpreting the Routh Table

- The no. of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

- A system is **Stable** if there are no sign changes in the first column of the Routh table.

Example :



Ref : N. S. Nise, Control Systems Engineering, 4th Ed. Wiley, 2004.

Example :

Solution: Since all the coefficients of the closed-loop characteristic equation $s^3 + 10s^2 + 31s + 1030$ are present, the system passes the Hurwitz test. So we must construct the Routh array in order to test the stability further.

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$

Example :

- it is clear that column 1 of the Routh array is: $\begin{pmatrix} 1 \\ 1 \\ -72 \\ 103 \end{pmatrix}$

- it has two sign changes (from 1 to -72 and from -72 to 103). Hence the system is ***unstable*** **with** ***two poles in the right-half plane*** **.**

Routh-Hurwitz Criterion: Special Cases

- The basic Routh table check fails in the following two cases:
  - Zero only in the first column of a row
  - Entire row consisting of zeros

- These cases need further analysis

Special Case – 1: Zero only in the first column

(1) Replace zero by ε.

Then let $\varepsilon \to 0$ either from left or right.

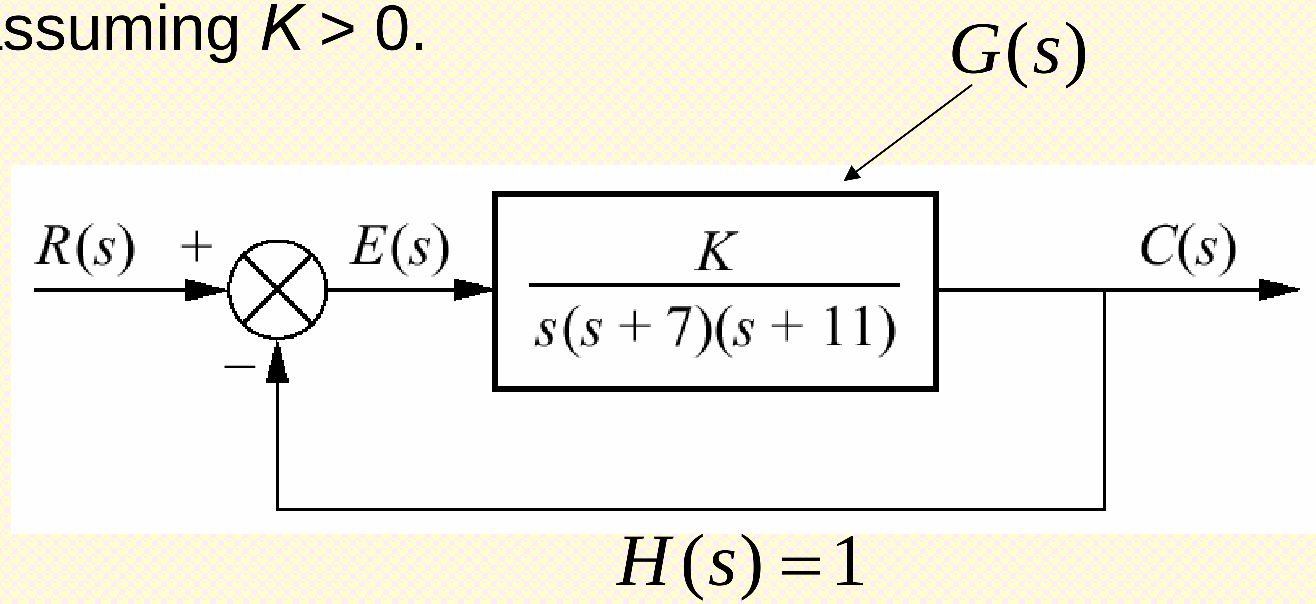
(2) Replace s by $(1/d)$. The resulting ploynomial will have roots which are reciprocal of the roots of the original polynomial. Hence they will have the same sign. The resulting ploynomial can be written by a polynomial with coefficient in reverse order.

Special Case – 2: Entire row that consists of zeros

- Form the Auxiliary equation from the row above the row of zero
- Differentiate the polynomial with respect to *s* and replace the row of zero by its Coefficients
- Continue with the construction of the Routh table and infer about the stability from the number of sign changes in the first column

Stabilizing control gain design via Routh-Hurwitz

Objective: To find the range of gain K for the following system to be stable, unstable and marginally stable, assuming $K > 0$.



Stability analysis via Routh-Hurwitz criterion

Closed loop transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^3 + 18s^2 + 77s + K}$$

Routh Table:

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Stability design via Routh-Hurwitz :

If $K < 1386$ system is stable (three poles in the LHS)

If $K > 1386$ system is unstable (two poles in RHS one in LHS)

If $K = 1386$ then an entire row will be zero.

Then the auxiliary equation is $P(s) = 18s^2 + 1386$

$$\frac{dp(s)}{ds} = 36s + 0$$

Routh table
(with *K* = 1386)

s^3	1	77
s^2	18	1386
s^1	~~0~~ 36	
s^0	1386	

The system is Stable (marginally stable)

Steady State Error Analysis

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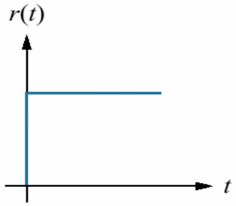
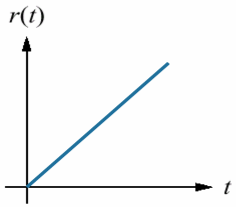
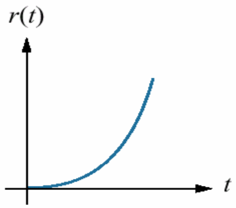
Definition and Test Inputs

- Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$

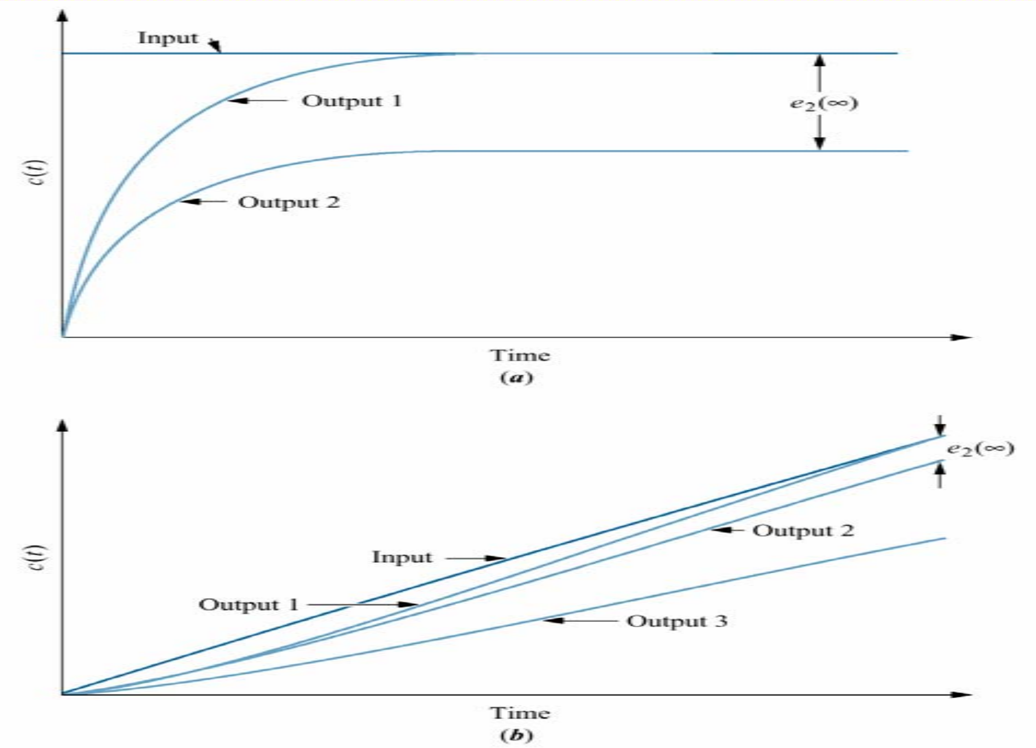
- Usual test inputs used for steady-state error analysis and design are Step, Ramp and Parabola inputs (justification comes from the Taylor series analysis).

Test Inputs

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Test waveforms for evaluating steady-state errors of control systems

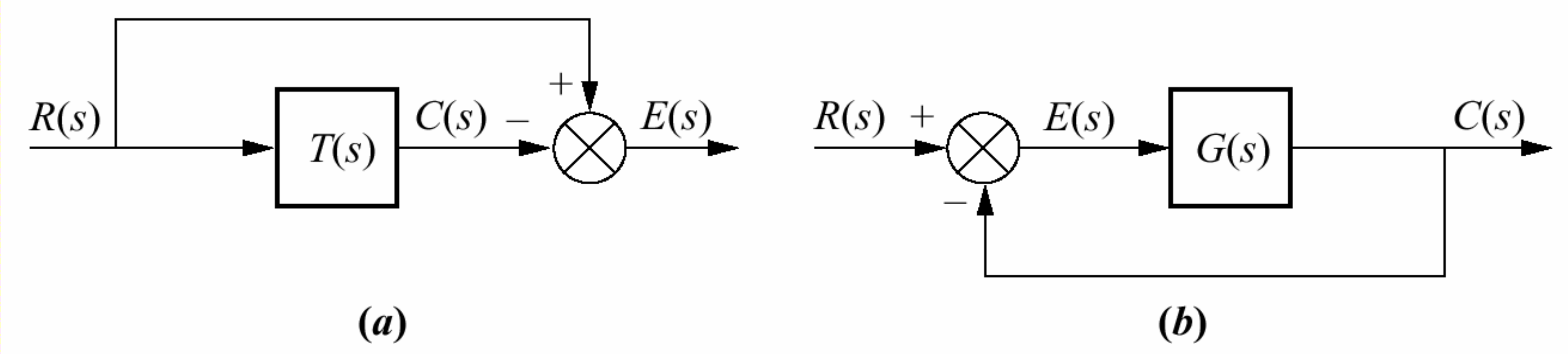
Evaluating Steady–State Errors



Ref: N. S. Nise, Control Systems Engineering, 4th Ed. Wiley, 2004.

Figure :
Steady-state error: **a.** step input **b.** ramp input

Evaluating Steady–State Errors



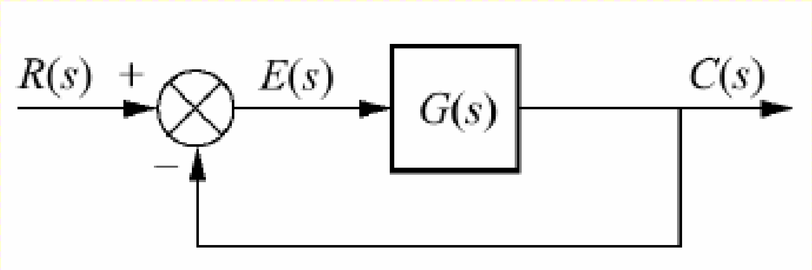
a. **General Representation**

b. **Representation for unity feedback systems**

Steady-State Error for Unity Feedback Systems

Steady-State Error in Terms of T(s),

$$E(s) = R(s) - C(s)$$
$$C(s) = R(s)T(s)$$
$$E(s) = R(s)[1 - T(s)]$$



Applying Final Value Theorem,

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$
$$e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)]$$

An Example

Problem :

Find the steady-state error for the system if

$T(s) = 5/(s^2 + 7s + 10)$ and the input is a unit step.

Solution : $R(s) = 1/s$ and $T(s) = 5/(s^2 + 7s + 10)$

This yields $E(s) = (s^2 + 7s + 5)/s(s^2 + 7s + 10)$

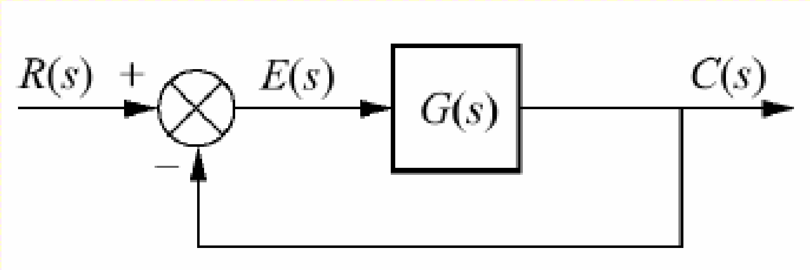
since $T(s)$ is stable, by final value

theorem , $e(\infty) = 1/2$

Steady-State Error for Unity Feedback Systems..

Steady State Error in Terms of G(s)

$$E(s) = R(s) - C(s)$$
$$C(s) = E(s)G(s)$$
$$E(s) = R(s) / \left(1 + G(s)\right)$$



By Final Value Theorem,

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Note: Input $R(s)$ and System $G(s)$ allow us to calculate steady-state error $e(\infty)$

Effect of input on steady-state error :

- Step Input :

$R(s) = 1/s$

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}$$

For zero steady-state error, $\lim_{s\to 0} G(s) = \infty$.

Hence $G(s)$ must have the form:

$$G(s) = \frac{(s+z_1)(s+z_2)\ldots}{s^n(s+p_1)(s+p_2)\ldots} \text{ and } n \geq 1$$

If $n = 0$, then the system will have finite steady state error.

Effect of input on steady-state error ..

- Ramp Input :

$$R(s) = 1/s^2$$

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

For zero steady-state error, $\lim_{s\to 0} sG(s) = \infty$.

Hence $G(s)$ must have the form:

$$G(s) = \frac{(s+z_1)(s+z_2)\ldots}{s^n(s+p_1)(s+p_2)\ldots} \quad and \;\; n \geq 2$$

Effect of input on steady-state error ..

- Parabolic input :

$R(s) = 1/s^3$

$$e(\infty) = e_{parabola}(\infty) = \lim_{s\to 0} \frac{s(1/s^3)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

For zero steady-state error, $\lim_{s\to 0} s^2 G(s) = \infty$

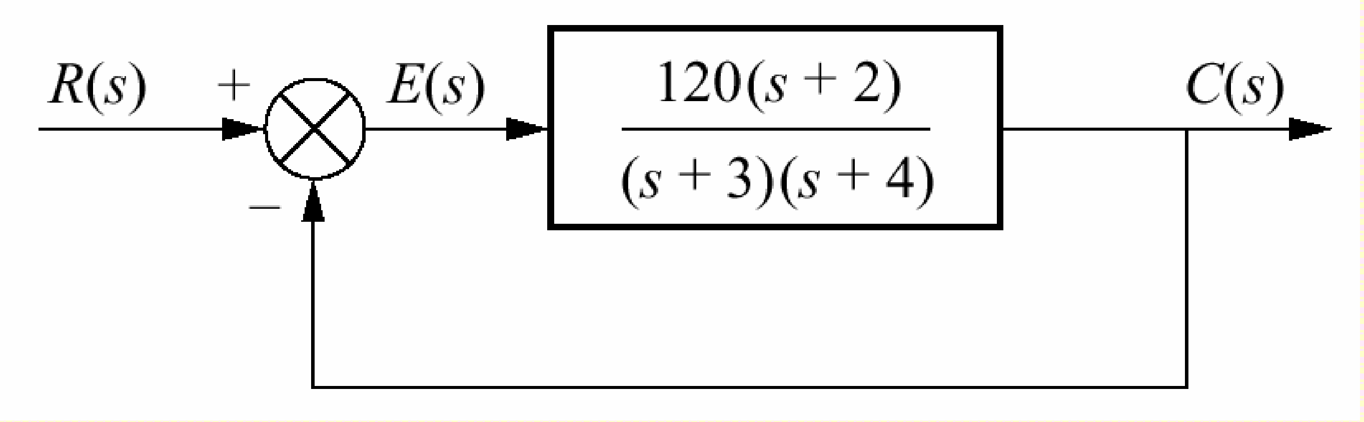
Hence $G(s)$ must have the form:

$$G(s) = \frac{(s+z_1)(s+z_2)\ldots}{s^n (s+p_1)(s+p_2)\ldots} \quad and \quad n \geq 3$$

An Example

Objective :

To find the steady-state errors of the following system for inputs of $5u(t)$, $5t\,u(t)$ and $5t^2u(t)$, where $u(t)$ is the unit step function.



Example

Solution :

First we verify that the closed loop system is stable.

Next, carryout the following analysis.

The Laplace transform of $5u(t)$, $5t\,u(t)$, $5t^2u(t)$ are $5/s, 5/s$ and $5/s^2$ respectively.

$$e(\infty) = e_{step}(\infty) = \frac{5}{1+\lim\limits_{s\to 0} G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim\limits_{s\to 0} sG(s)} = \frac{5}{0} = \infty$$

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim\limits_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty.$$

Static Error Constants and System Type :

The steady-state error performance specifications are called "static error constants", defined as follows:

Position constant $K_p = \lim_{s \to 0} G(s)$

Velocity constant $K_v = \lim_{s \to 0} sG(s)$

Acceleration constant $K_a = \lim_{s \to 0} s^2 G(s)$

"System Type" is the value of n in the denominator of $G(s)$;

i.e. Number of pure integrators in the forward path.

Note: $n = 0, 1, 2$ indicates Type $0, 1, 2$ system respectively.

Relationship between input, system type, static error constants and steady state errors

Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Ref: N.S.Nise, Control Systems Engineering,4th Ed. Wiley,2004

Interpreting the steady-state error specification :

Question : What information is contained in $K_p = 1000$?

Answer :

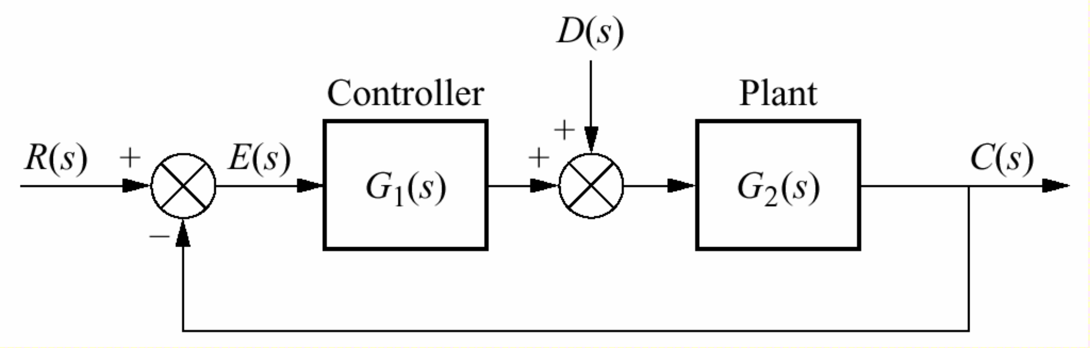
The system is stable.

The system is Type 0, since K_p is finite.

If the input signal is unit step, then

$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{1+1000} = \frac{1}{1001}$$

Steady-State Error for Disturbance Inputs



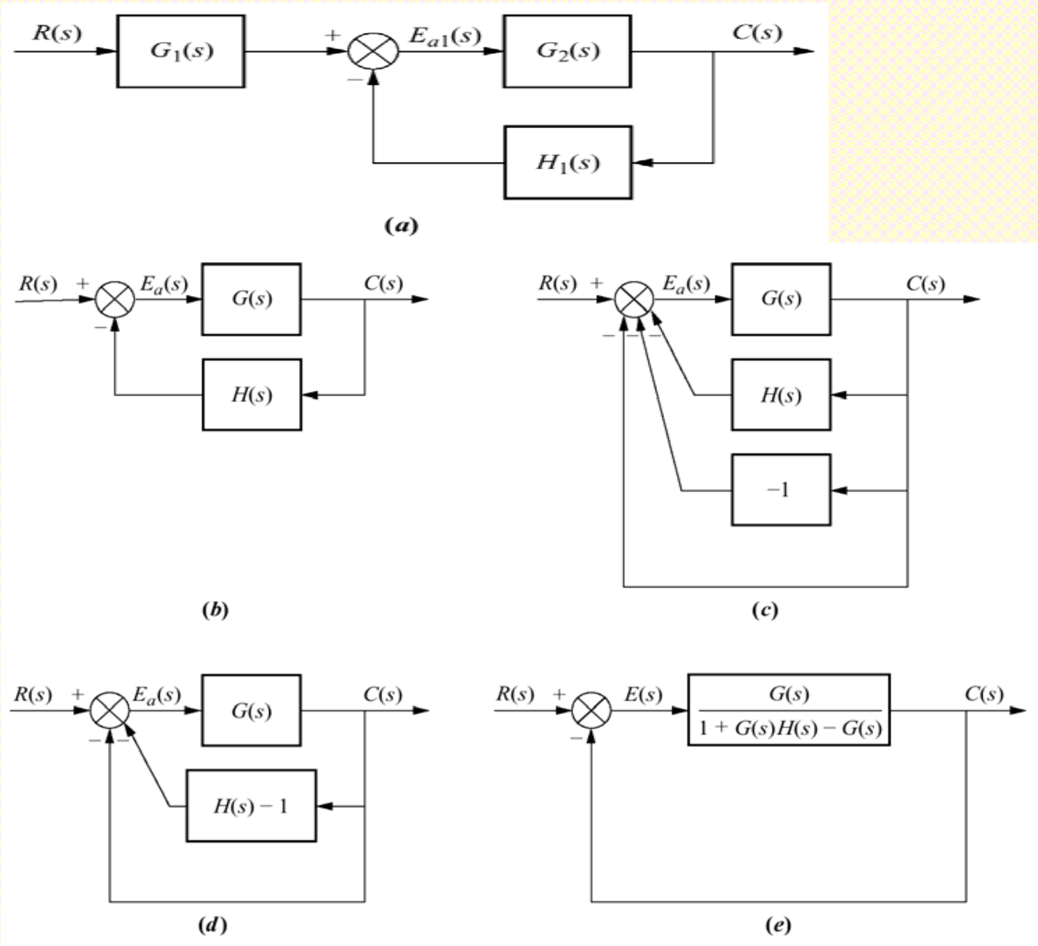
$$E(s) = R(s) - C(s) = R(s) - \left[E(s)G_1(s)G_2(s) + D(s)G_2(s)\right]$$

$$\left[1 + G_1(s)G_2(s)\right]E(s) = R(s) - D(s)G_2(s)$$

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s) = E_R(s) + E_D(s)$$

Hence the steady state error $E_D(s)$ can be reduced by either increasing $G_1(s)$ or decreasing $G_2(s)$.

Steady-State Error for Non-unity Feedback Systems



Forming an Equivalent Unity Feedback System

Sensitivity

- The degree to which changes in system parameters affect system transfer functions, and hence performance, is called sensitivity.

- The greater the sensitivity, the less desirable the effect of a parameter change.

Sensitivity

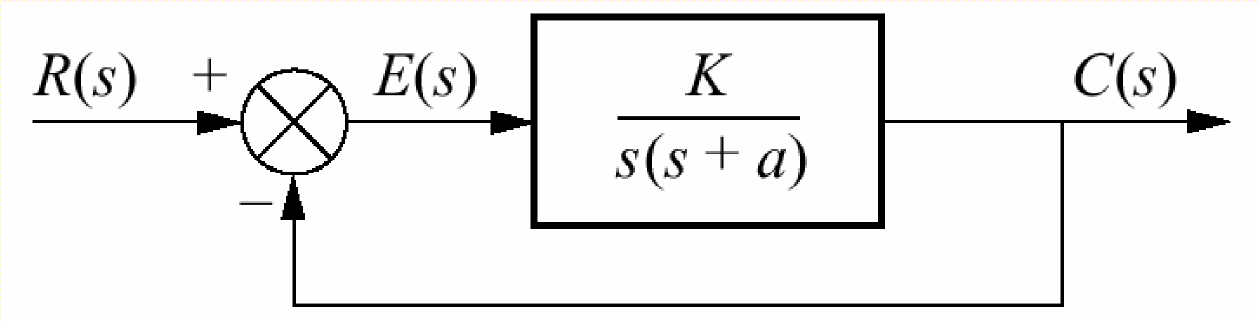
$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function } F}{\text{Fractional change in the parameter } P}$$

$$= \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P}$$

$$= \lim_{\Delta P \to 0} \frac{P}{F}\left(\frac{\Delta F}{\Delta P}\right) = \frac{P}{F}\left(\frac{\partial F}{\partial P}\right)$$

Note : In some cases feedback reduces the sensitivity of a system's steady-state error to changes in system parameters.

Example



Ramp input: $R(s) = 1/s^2$

$$K_v = \lim_{s \to 0} sG(s) = \frac{K}{a}$$

$$e_\infty\Big|_{ramp} = \frac{1}{K_v} = \frac{a}{K}$$

$$S_{e:a} = \frac{a}{e}\left(\frac{\partial e}{\partial a}\right) = \frac{a}{a/K}\left(\frac{1}{K}\right) = 1$$

$$S_{e:K} = \frac{K}{e}\left(\frac{\partial e}{\partial K}\right) = \frac{K}{a/K}\left(\frac{-a}{K^2}\right) = -1$$

Thanks for the Attention…!