Lecture – 34

Stability Analysis of Nonlinear Systems Using Lyapunov Theory – II

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Outline

- Construction of Lyapunov Functions
- Definitions
  - Invariant Sets
  - Limit Sets
- LaSalle's Theorem

Construction of Lyapunov Functions

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Variable Gradient Method:

* Select a $\nabla V = \dfrac{\partial V}{\partial X} = g(X)$ that contains some adjustable parameters

* Then $dV(X) = \left(\dfrac{\partial V}{\partial X}\right)^T dX$

$$\int_{\tilde{X}=0}^{X} dV(\tilde{X}) = \int_{\tilde{X}=0}^{X} \left(\frac{\partial V}{\partial \tilde{X}}\right)^T d\tilde{X}$$

$$V(X) - V(0) = \int_{\tilde{X}=0}^{X} g(\tilde{X})\, d\tilde{X}$$

> Note:
> To recover a unique V, $\nabla V = g(X)$ must satisfy the "Curl Condition":
> *i.e.* $\dfrac{\partial g_i}{\partial x_j} = \dfrac{\partial g_j}{\partial x_i}$

However, note that the intergal value depends on the initial and final states (not on the path followed). Hence, integration can be conveniently done along each of the co-ordinate axes in turn; i.e.

Variable Gradient Method:

$$V(X) = \int_0^{x_1} g_1(\tilde{x}_1, 0, \ldots\ldots, 0)\, d\tilde{x}_1$$

$$+ \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\, d\tilde{x}_2$$

$$\vdots$$

$$+ \int_0^{x_n} g_n(x_1, \ldots\ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n$$

Note: The free parameter of $g(X)$ are constrained to satisfy the symmetric condition, which is satisfied by all gradients of a scalar functions.

Variable Gradient Method:

<u>Theorem</u>: A function $g(X)$ is the gradient of a scalar function $V(X)$ <u>if and only if</u> the matrix $\left[\dfrac{\partial g(X)}{\partial X}\right]$ is symmetric; where

$$\left[\frac{\partial g(X)}{\partial X}\right] \triangleq \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial x_1} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{bmatrix}$$

Proof: Please see Marquez book (Appendix)

Variable Gradient Method:

<u>Proof</u> : (Necessity)

Assume: $$g(X) = \frac{\partial V}{\partial X}$$

$$\frac{\partial g(X)}{\partial X} = \frac{\partial^2 V}{\partial X^2}$$

$$= \begin{pmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{pmatrix}$$

Variable Gradient Method:

$$\because \qquad \frac{\partial^2 V}{\partial x_1 \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} \quad \Rightarrow \quad \boxed{\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}}$$

Hence,the matrix $\left[\dfrac{\partial g\left(X\right)}{\partial X}\right]$ should be symmetric.

Variable Gradient Method:

<u>Sufficiency</u>: Assume $$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$$

$$\left[\text{To show } \frac{\partial V}{\partial x_i} = g_i\left(X\right) \quad \forall i \right]$$

Variable Gradient Method:

We have:

$$V(X) = \int_0^x g(\tilde{x})\, d\tilde{x}$$

$$= \int_0^{x_1} g_1(\tilde{x}_1, 0, ..., 0)\, d\tilde{x}_1$$

$$+ \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, ..., 0)\, d\tilde{x}_2$$

$$+ \int_0^{x_n} g_n(x_1, x_2, ..., x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n$$

Variable Gradient Method:

$$
\begin{aligned}
\frac{\partial V}{\partial x_1} &= g_1(x_1, 0, \ldots\ldots, 0) \\
&\quad + \int_0^{x_2} \frac{\partial g_2}{\partial x_1}(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\, d\tilde{x}_2 \\
&\quad + \int_0^{x_n} \frac{\partial g_n}{\partial x_1}(x_1, x_2, \ldots\ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n \\
&= g_1(x_1, 0, \ldots\ldots, 0) + \int_0^{x_2} \frac{\partial g_1}{\partial x_2}(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\, d\tilde{x}_2 + \ldots \\
&\quad + \int_0^{x_n} \frac{\partial g_1}{\partial x_n}(x_1, x_2, \ldots\ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n
\end{aligned}
$$

Variable Gradient Method:

$$= \quad g_1(x_1,0,......,0) + g_1(x_1,\tilde{x}_2,0,......,0)\Big|_{\tilde{x}_2=0}^{x_2}$$

$$+\cdots+ g_1(x_1,x_2,......,x_{n-1},\tilde{x}_n)\Big|_{\tilde{x}_n=0}^{x_n}$$

$$= \quad g_1(x_1,0,......,0) + \left[g_1(x_1,\tilde{x}_2,0,......,0) - g_1(x_1,0,......,0)\right]$$

$$+\cdots+\left[g_1(x_1,x_2,......,x_n) - g_n(x_1,x_2,......,x_n,0)\right]$$

$$= \quad g_1(x_1,x_2,......,x_n)$$

i.e $$\boxed{\frac{\partial V}{\partial x_1} = g_1(X)}$$

Similarly $$\frac{\partial V}{\partial x_i} = g_i(X) \quad , \quad \forall i = 1,\cdots,n$$

Variable Gradient Method: Example

Problem: Analyze the stability behaviour of the following system

$$\dot{x}_1 = -a x_1$$

$$\dot{x}_2 = b x_2 + x_1 x_2^2$$

Solution: $X = 0$ is an equilibrium point

$$\text{Assume } \frac{\partial V}{\partial X} = g(X) = \underbrace{\begin{pmatrix} k_1 & k \\ k & k_2 \end{pmatrix}}_{\text{A symmetric matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\text{Note:} \quad \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} = k \right)$$

Variable Gradient Method: Example

Further, let us assume

$$\therefore \qquad \frac{\partial V}{\partial X} = \begin{bmatrix} g_1(X) \\ g_2(X) \end{bmatrix} = \begin{bmatrix} k_1 x_1 \\ k_2 x_2 \end{bmatrix}$$

$$\Rightarrow \quad V(X) = \int_0^{x_1} g_1(\tilde{x}_1, 0)\, d\tilde{x}_1 + \int_0^{x_2} g_2(x_1, \tilde{x}_2)\, d\tilde{x}_2$$

$$= \int_0^{x_1} k_1 \tilde{x}_1 d\tilde{x}_1 + \int_0^{x_2} k_2 \tilde{x}_2 d\tilde{x}_2$$

$$= \frac{1}{2}\left(k_1 {x_1}^2 + k_2 {x_2}^2\right)$$

Variable Gradient Method:

Choose $\boxed{k_1, k_2 > 0}$

Then $V(X) > 0 \quad \forall X \neq 0$ and $V(0) = 0$

$V(X)$ is a Lyapunov function candidate.

$$\dot{V}(X) = g^T(X)\, f(X) = \begin{bmatrix} k_1 x_1 & k_2 x_2 \end{bmatrix} \begin{bmatrix} -a x_1 \\ b x_2 + x_1 x_2{}^2 \end{bmatrix}$$

$$= -k_1 a x_1^2 + k_2 (b + x_1 x_2) x_2^2$$

Let us choose $k_1 = k_2 = 1$. Then

$$\dot{V}(X) = -a x_1^2 + (b + x_1 x_2) x_2^2$$

Variable Gradient Method:

Unless we know about a, b at this point nothing can be said about $\dot{V}(X)$. Let us assume $a > 0,\ b < 0.$ Then

$$\dot{V}(X) = -a x_1^{\,2} - \underbrace{\left(|b| - x_1 x_2\right)}_{>0 \text{ (for small } x_1 x_2\text{)}} x_2^2$$

$\therefore \quad \dot{V}(X) < 0$ in some domain $D \subset \mathbb{R}^2$ and $0 \in D$

i.e $\dot{V}(X)$ is negative definite in D

$\therefore$ The system is locally asymptotically stable!

Krasovskii's Method

Let us consider the system $\dot{X} = f(X)$

Let $A(X) \triangleq \left[\frac{\partial f}{\partial X}\right]$: Jacobian matrix

Theorem :

If the matrix $F(X) \triangleq A(X) + A^T(X)$ is ndf for all $X \in D \quad (0 \in D)$,

then the equilibrium point is locally asymptotically stable and a Lyapunov function for the system is

$$V(X) = f^T(X) f(X)$$

Note: If $D = \mathbb{R}^n$ and $V(X)$ is radially unbounded,

then the equilibrium point is globally asymptotically stable.

Krasovskii's Method

Claim-1: Since $F(X)$ is ndf, $A(X)$ is invertible.

Proof (by contradiction):

Let $A(X)$ be singular

Then $\exists\, Y_0 \neq 0 :\ A(X)Y_0 = 0$

But $\quad Y_0^{\,T} F\, Y_0 = Y_0^{\,T}\left(A + A^T\right)Y_0$

$$= Y_0^{\,T}\left(AY_0\right) + \left(Y_0^{\,T} A^T\right)Y_0 = 0$$

i.e. $\quad F$ is not ndf.

Hence, $A(X)$ is non-singular (i.e., it is invertible).

Krasovskii's Method

Claim-2: The invertibility (and continuity) of $A(X)$ guarantees that the function $f(X)$ can be uniquely inverted.

Justification:

This is perphaps straight forward from unifrom convergence property of Taylor series expansion.

This leads to the conclusion that the dynamic system has only one equilibrum point in D. i.e. $f(X) \neq 0, \quad \forall X \neq 0, X \in D.$

$\therefore \quad V(X) = f^T(X) f(X)$ is pdf.

Krasovskii's Method

$$\dot{V}(X) = f^T \dot{f} + \dot{f}^T f$$

$$= f^T \left[\frac{\partial f}{\partial X}\right]^T \dot{X} + \dot{X}^T \left[\frac{\partial f}{\partial X}\right] f$$

$$= f^T \left(A^T + A\right) f$$

$$= f^T F \, f$$

Hence, if $F(X)$ is negative definite, $\dot{V}(X)$ is ndf.

So, by Lyapunov's theorem, $X = 0$ is asymptotically stable.

Krasovskii's Method

Note: The global asymptotic stability of the system is guaranteed by the Global version of Lyapunov's direct method.

Comment: While the usage of this result is fairly straight forward, its applicability is limited in practice since $F\left(X\right)$ for many systems do not satisfy the negative definite property.

Generalized Krasovskii's Theorem

Theorem :

Let $$A(X) \triangleq \left[\frac{\partial f(X)}{\partial X}\right]$$

A sufficent condition for the origin to be asymptotically stable is that $\exists$ two pdf matrices P and Q: $\forall X \neq 0$, the matrix

$$F(X) = A^T P + PA + Q$$

is negative semi-definite in some neighbourhood D of the origin.

In addition, if $D = \mathbb{R}^n$ and $V(X) \triangleq f^T(X)\, P\, f(X)$ is radially unbounded, then the system is globally asymptotically stable.

Generalized Krasovskii's Theorem

Proof :

$$V(X) = f^T(X)Pf(X)$$

$$\dot{V}(X) = \left[f^T P\ \dot{f} + \dot{f}^T P\ f\right]$$

$$= f^T P \left(\frac{\partial f}{\partial X}\right)^T \dot{X} + \left[\left(\frac{\partial f}{\partial X}\right)^T \dot{X}\right]^T Pf$$

$$= f^T P A^T f + f^T AP\ f$$

$$= f^T\left(PA^T + AP + Q - Q\right) f$$

$$= \underbrace{f^T\left(PA^T + AP + Q\right) f}_{nsdf} - \underbrace{f^T Q f}_{ndf}$$

< 0 (ndf) Hence, the result.

Example

Problem: Analyze the stability behaviour of the following system

$$\dot{x}_1 = -6x_1 + 2x_2$$
$$\dot{x}_2 = \quad 2x_1 - 6x_2 - 2x_2^{\,3}$$

Solution:

$$A = \left[\frac{\partial f}{\partial X}\right] = \begin{bmatrix} -6 & 2 \\ 2 & -6-6x_2^{\,2} \end{bmatrix}$$

$$F = A + A^T = \begin{bmatrix} -12 & 4 \\ 4 & -12-12x_2^{\,2} \end{bmatrix}$$

Example

Eigen values of F:

$$\begin{vmatrix} \lambda + 12 & -4 \\ -4 & \lambda + 12 + 12{x_2}^2 \end{vmatrix} = 0$$

$$(\lambda + 12)^2 + (\lambda + 12)12{x_2}^2 - 16 = 0$$

$$\lambda^2 + 24\lambda + 144 + 12{x_2}^2\lambda + 144{x_2}^2 - 16 = 0$$

$$\lambda^2 + \left(24 + 12{x_2}^2\right)\lambda + \left(128 + 144{x_2}^2\right) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[-\left(24 + 12{x_2}^2\right) \pm \sqrt{\left(24 + 12{x_2}^2\right)^2 - 4\left(128 + 144{x_2}^2\right)}\right]$$

Example

$$= -(12+6x_2{}^2) \pm \underbrace{\sqrt{\left(12+6x_2{}^2\right)^2 - \left(128+144x_2{}^2\right)}}_{0<(*)<(12+6x_2^2)}$$

$$< 0 \qquad \forall\ x_2 \in \mathbb{R}$$

$$\therefore \quad \text{A is ndf in } \mathbb{R}^2$$

Morever, $V\left(X\right) = f^T\left(X\right) f\left(X\right)$

$$= \left(-6x_1 + 2x_2\right)^2 + \left(2x_1 - 6x_2 - 2x_2{}^3\right)^2$$

$$\rightarrow \infty \ \text{ as } \ \left\|X\right\| \rightarrow \infty$$

$\therefore \quad X = 0$ is <u>globally asymptotically stable.</u>

Invariant Sets & La Salle's Theorem

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Invariant Set

A set M is said to be an "invariant set" with respect to the system $\dot{X} = f(X)$ if:

$$\boxed{X(0) \in M} \quad \Rightarrow \quad \boxed{X(t) \in M \,, \forall t > 0}$$

Examples:

(i) An equilibrium point $(M = X_e)$

(ii) Any trajectary of an autonomous system $\left(M = X(t)\right)$

Invariant Set

(iii) A limit cycle

(iv) $M = \mathbb{R}^n$

(v) $\Omega_l = \left\{ \begin{array}{l} X \in \mathbb{R}^n : V(X) \le l \\ \text{where, } V(X) \text{ is a continuously differentiable function} \\ \text{such that } \dot{V}(X) \le 0 \text{ along the solution of } \dot{X} = f(X) \end{array} \right\}$

Note: (1) $V(X)$ need not be pdf.

(2) The condition implies that once the trajectory crosses the surface $V(X) = c,$ it can never come out again.

Limit Set

Definition:

Let $X(t)$ be a trajectory of the dynamical system $\dot{X} = f(X)$ .Then the set N is called the limit set (or positive limit set) of $X(t)$ if for any $p \in N$, $\exists$ a sequence of times $\{t_n\} \in [0,\infty]$ such that $X(t_n) \to p$ as $t_n \to \infty$.

Note: Roughly, the limit set N of $X(t)$ is whatever $X(t)$ tends to in the limit.

Limit Set

Example:

(i) An asymptotically stable equilibrium point is the limit set of any solution starting from a close neighbourhood of the equilibrium point.

(ii) A stable limit cycle is the limit set for any solution starting sufficiently close to it

Some Useful Results

Lemma-1:

If the solution $X\left(t, t_0, X_0\right)$ of the system $\dot{X} = f\left(X\right)$ is bounded for $t > t_0$,then its limit set N is:

$$\left.\begin{array}{l}\text{(i) bounded}\\ \text{(ii) closed}\\ \text{(iii) Non-empty}\end{array}\right\}\left(\text{i.e. it is a non empty "compact set"}\right)$$

Moreover, as $t \to \infty$, the solution approaches N.

Lemma-2: The limit set N of a solution $X\left(t, t_0, X_0\right)$ of the autonomous system $\dot{X} = f\left(X\right)$ is invariant with respect to the same system.

A Useful Theorem (Subset of LaSalle's Theorem)

Theorem : The equilibrium point $X = 0$ of the autonomous system $\dot{X} = f(X)$

is asymptotically stable if:

(i) $V(X) > 0$ (pdf) $\forall X \in D \quad [0 \in D]$

(ii) $\dot{V}(X) \leq 0$ (nsdf) in a bounded region $R \subset D$

(iii) $\dot{V}(X)$ does not vanish along any trajectory in R

other than the null solution $X = 0$

Morever,

If the above conditions hold good for R = $\mathbb{R}^n$ and $V(X)$ is radially unbounded,

then $X = 0$ is globally asymptotically stable.

Proof of the Theorem

$$\dot{V} \leq 0$$

$\Rightarrow$ The system is stable

i.e. for each $\varepsilon > 0$, $\exists \delta > 0$:

$$\left\| X_0 \right\| < \delta \Rightarrow \left\| X(\mathrm{t}) \right\| < \varepsilon$$

or, Any solution starting inside the closed ball B_δ

will remain within the closed ball B_ε

$\Rightarrow$ The solution (starting within B_δ) is bounded.

Proof of the Theorem

Hence, $X(t)$ tends to its limit set $N \subset B_\varepsilon$

and B_ε is compact. (By Lemma - 1)

Moreover, $V(X)$ is continous on the compact set B_ε

and $\dot{V}(X) \leq 0$, $\therefore$ $V(X) \rightarrow L \geq 0$ as $t \rightarrow \infty$

i.e $V(X) = L \quad \forall X \in N$ (N: the limit set)

Note that N is invariant set with respect to the system $\dot{X} = f(X)$ (By Lemma - 2)

Proof of the Theorem

$\Rightarrow$ Any solution that starts in N will remain within it for all furture time.

However, along that solution $\dot{V}(X) = 0$, as $V(X) = L$

But, by the assumption of the theorem, $\dot{V}(X)$ does not vanish along any trajectory other than the null solution $X = 0$

Hence, Any solution starting in $R \subset B_\delta$ converges to $X = 0$ as $t \to \infty$

Example – 1: Pendulum with Friction

Example: (Pendulum with friction)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \left(\frac{k}{m}\right)x_2$$

$$V(X) = \frac{1}{2}ml^2 x^2{}_2 + mgl\left(1 - \cos x_1\right)$$

$$> 0 \;\; \forall X \in D \;\; = \left(-\pi, \pi\right) \times R$$

$$\dot{V}(X) = -kl^2 x^2{}_2 : \text{ nsdf}$$ [Note: $0 \in$ D]

Example – 1: Pendulum with Friction

Now let us examine the condition

$$\dot{V}\left(X\right)=0 \quad \forall t$$

$$-kl^2 x^2{}_2 = 0$$

$$\Leftrightarrow \qquad x_2 = 0 \quad \forall t \qquad \Rightarrow \quad \dot{x}_2 = 0. \text{ Hence}$$

$$\frac{g}{l}\sin x_1 + \frac{k}{m}x_2 = 0$$

$$\sin x_1 = 0 \quad \left(\because \ x_2 = 0\right) \quad \Rightarrow \quad x_1 = 0 \quad \left[\text{as } \ x_1 \in \left(-\pi, \pi\right)\right]$$

Hence, $\dot{V}\left(X\right)$ happens only for $X = 0$.

Hence, $X = 0$ is <u>locally asymptotically stable</u>!

Example – 2

Example:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2$$

Solution: Let $V(X) = \alpha x^2{}_1 + x_2{}^2 \;,\; \alpha > 0$

$$\dot{V}(X) = \left(\frac{\partial V}{\partial X}\right)^T f(X)$$

$$= \begin{bmatrix} 2\alpha x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 \end{bmatrix}$$

$$= 2\alpha x_1 x_2 - 2x_2^2 - 2\alpha x_1 x_2 - 2(x_1 + x_2)^2 x_2^2$$

Example – 2

$$\dot{V}(X) = -2{x_2}^2\left[1+(x_1+x_2)^2\right]$$

$$\leq 0 \quad \text{(nsdf)}$$

Now $\dot{V}(X) = 0 \quad \forall t$

$$\Leftrightarrow \quad x_2(t) = 0 \quad \forall t$$

$$\Rightarrow \quad \dot{x}_2 = 0$$

$$-x_2 - \alpha x_1 - (x_1+x_2)^2 x_2 = 0 \quad (\text{However, } x_2 = 0)$$

$$\therefore \quad x_1 = 0 \qquad \text{i.e.} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example – 2

Here we have :

(i) $\dot{V}(X)$ does not vanish along any trajectory other than $X = 0$

(ii) $\dot{V} \leq 0$ in $\mathbb{R}^n$

(iii) $V(X)$ is radially unbounded,

Hence, the origin is Globally asymptotically stable.

LaSalle's Theorem

Let $V: D \to \mathbb{R}$ be a continuously differentiable (not necessarily pdf) function

and (i) $M \subset D$ be a compact set, which is invariant with respect to the solution of $\dot{X} = f(X)$

(ii) $\dot{V} \le 0$ in M

(iii) $E = \{X : X \in M \text{ and } \dot{V}(X) = 0\}$

i.e. E is the set of all points of $M : \dot{V} = 0$

(iv) N is the largest invariant set in E

Then Every solution starting in M approaches N as $t \to \infty$.

Lasalle’s Theorem

Remarks:

(i) $V(X)$ is required only to be continuously differentiable

It need not be positive definite.

(ii) LaSalle's Theorem applies not only to equilibrium

points, but also to more general dynamic

behaviours such as limit cycles.

(iii) The earlier theorems (on asymptotic stability) can

be derived as a corollary of this theorem.

References

- H. J. Marquez: *Nonlinear Control Systems Analysis and Design*, Wiley, 2003.
- J-J. E. Slotine and W. Li: *Applied Nonlinear Control*, Prentice Hall, 1991.
- H. K. Khalil: *Nonlinear Systems*, Prentice Hall, 1996.

Thanks for the Attention...!