Lecture – 33

Stability Analysis of Nonlinear Systems Using Lyapunov Theory – I

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Outline

- Motivation
- Definitions
- Lyapunov Stability Theorems
- Analysis of LTI System Stability
- Instability Theorem
- Examples

References

- H. J. Marquez: *Nonlinear Control Systems Analysis and Design*, Wiley, 2003.
- J-J. E. Slotine and W. Li: *Applied Nonlinear Control*, Prentice Hall, 1991.
- H. K. Khalil: *Nonlinear Systems*, Prentice Hall, 1996.

Techniques of Nonlinear Control Systems Analysis and Design

- Phase plane analysis
- Differential geometry (Feedback linearization)
- Lyapunov theory
- Intelligent techniques: Neural networks, Fuzzy logic, Genetic algorithm etc.
- Describing functions
- Optimization theory (variational optimization, dynamic programming etc.)

Motivation

- Eigenvalue analysis concept does not hold good for nonlinear systems.
- Nonlinear systems can have multiple equilibrium points and limit cycles.
- Stability behaviour of nonlinear systems need not be always global (unlike linear systems).
- Need of a systematic approach that can be exploited for control design as well.

Definitions

System Dynamics

$\dot{X} = f(X) \qquad f : D \to \mathbb{R}^n$ (a locally Lipschitz map)

$D :$ an open and connected subset of $\mathbb{R}^n$

Equilibrium Point (X_e)

$$\dot{X}_e = f(X_e) = 0$$

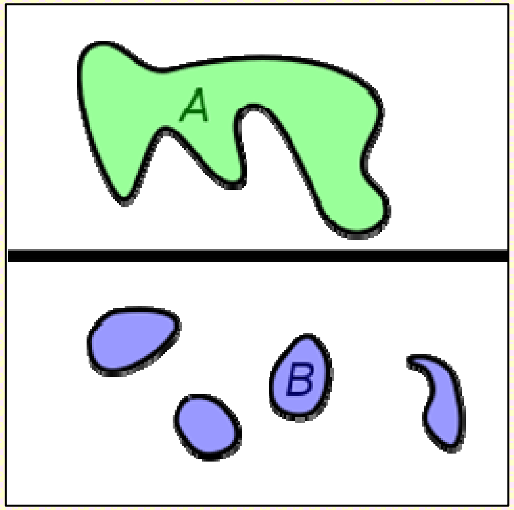
Definitions

Open Set A set $A \subset \mathbb{R}^n$ is open if

for every $p \in A,\ \exists\ B_r(p) \subset A$

Connected Set

- A **connected set** is a set which cannot be represented as the union of two or more disjoint nonempty open subsets.
- Intuitively, a set with only one piece.



Space *A* is connected, *B* is not.

Definitions

Stable Equilibrium

X_e is stable, provided for each $\varepsilon > 0, \; \exists \delta(\varepsilon) > 0$:

$$\left\| X(0) - X_e \right\| < \delta(\varepsilon) \quad \Rightarrow \quad \left\| X(t) - X_e \right\| < \varepsilon \quad \forall t \geq t_0$$

Unstable Equilibrium

If the above condition is not satisfied, then the equilibrium point is said to be unstable

Definitions

Convergent Equilibrium

$$\text{If } \exists\delta: \quad \left\| X(0) - X_e \right\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} X(t) = X_e$$

Asymptotically Stable

If an equilibrium point is both stable and convergent, then it is said to be asymptotically stable.

Definitions

Exponentially Stable

$$\exists\, \alpha, \lambda > 0 : \quad \left\| X(t) - X_e \right\| \le \alpha \left\| X(0) - X_e \right\| e^{-\lambda t} \quad \forall t > 0$$

$$\text{whenever} \quad \left\| X(0) - X_e \right\| < \delta$$

Convention

The equilibrium point $X_e = 0$

(without loss of generality)

Definitions

A function $V : D \to \mathbb{R}$ is said to be **positive semi definite** in D if it satisfies the following conditions:

$$(i)\, 0 \in D \;\; and \;\; V(0) = 0$$

$$(ii)\, V\left(X\right) \geq 0, \;\; \forall X \in D$$

$V : D \to \mathbb{R}$ is said to be **positive definite in** D if condition (*ii*) is replaced by $V\left(X\right) > 0 \;\; in \;\; D - \{0\}$

$V : D \to \mathbb{R}$ is said to be **negative definite (semi definite)** in D if $-V\left(X\right)$ is positive definite.

Lyapunov Stability Theorems

Theorem – 1 (Stability)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X), \; f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that:

$(i) \quad V(0) = 0$

$(ii) \; V(X) > 0, \; in \; D - \{0\}$

$(iii) \; \dot{V}(X) \le 0, \; in \; D - \{0\}$

Then $X = 0$ is "stable".

Lyapunov Stability Theorems

Theorem – 2 (Asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f\left(X\right), \;\; f : D \rightarrow \mathbb{R}^n$.

Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function

such that:

$(i) \quad V\left(0\right) = 0$

$(ii) \; V\left(X\right) > 0, \;\; in \; D - \{0\}$

$(iii) \dot{V}\left(X\right) < 0, \;\; in \; D - \{0\}$

Then $X = 0$ is "asymptotically stable".

Lyapunov Stability Theorems

Theorem – 3 (Globally asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f\left(X\right),\ \ f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

$(i)\ \ V\left(0\right) = 0$

$(ii)\ \ V\left(X\right) > 0,\ \ in\ D - \{0\}$

$(iii)\ V\left(X\right)$ is "radially unbounded"

$(iv)\ \dot{V}\left(X\right) < 0,\ \ in\ D - \{0\}$

Then $X = 0$ is "globally asymptotically stable".

Lyapunov Stability Theorems

Theorem – 3 (Exponentially stable)

Suppose all conditions for asymptotic stability are satisfied.

In addition to it, suppose $\exists$ constants k_1, k_2, k_3, p :

$(i)\quad k_1 \|X\|^p \le V(X) \le k_2 \|X\|^p$

$(ii)\quad \dot{V}(X) \le -k_3 \|X\|^p$

Then the origin $X = 0$ is "exponentially stable".

Moreover, if these conditions hold globally, then the origin $X = 0$ is "globally exponentially stable".

Example:
Pendulum Without Friction

- System dynamics

$$x_1 \triangleq \theta, \;\; x_2 \triangleq \dot{\theta} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\left(g/l\right)\sin x_1 \end{bmatrix}$$

- Lyapunov function

$$\begin{aligned} V &= KE + PE \\ &= \frac{1}{2} m\left(\omega l\right)^2 + mgh \\ &= \frac{1}{2} ml^2 x_2^2 + mg\left(1 - \cos x_1\right) \end{aligned}$$

Pendulum Without Friction

$$\dot{V}(X) = (\nabla V)^T \; f(X)$$

$$= \begin{bmatrix} \dfrac{\partial V}{\partial x_1} & \dfrac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) & f_2(X) \end{bmatrix}^T$$

$$= \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 & -\dfrac{g}{l} \sin x_1 \end{bmatrix}^T$$

$$= mglx_2 \sin x_1 - mglx_2 \sin x_1 = 0$$

$$\dot{V}(X) \leq 0 \quad (\text{nsdf})$$

Hence, it is a "stable" system.

Pendulum With Friction

Modify the previous example by adding the friction force $kl\dot{\theta}$

$$ma = -mg\sin\theta - kl\dot{\theta}$$

Defining the same state variables as above

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

Pendulum With Friction

$$\dot{V}(X) = (\nabla V)^T f(X)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) & f_2(X) \end{bmatrix}^T$$

$$= \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 & -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}^T$$

$$= -kl^2 x_2^2$$

$$\dot{V}(X) \leq 0 \quad (\text{nsdf})$$

Hence, it is also just a "stable" system.
(A frustrating result..!)

Analysis of Linear Time Invariant System

System dynamics: $\dot{X} = AX, \quad A \in \mathbb{R}^{n\times n}$

Lyapunov function: $V(X) = X^T P X, \qquad P > 0 \ (\text{pdf})$

Derivative analysis:
$$\dot{V} = \dot{X}^T P X + X^T P \dot{X}$$
$$= X^T A^T P X + X^T P A X$$
$$= X^T \left(A^T P + PA\right) X$$

Analysis of Linear Time Invariant System

For stability, we aim for $\dot{V} = -X^T Q X \quad (Q > 0)$

By comparing $X^T \left(A^T P + PA \right) X = -X^T Q X$

For a non-trivial solution

$$PA + A^T P + Q = 0$$

(Lyapunov Equation)

Analysis of Linear Time Invariant System

Theorem : The eigenvalues λ_i of a matrix $A \in \mathbb{R}^{n \times n}$ satisfy $\text{Re}(\lambda_i) < 0$ if and only if for any given symmetric *pdf* matrix Q, $\exists$ a unique *pdf* matrix P satisfying the Lyapunov equation.

Proof: Please see Marquez book, pp.98-99.

Note : P and Q are related to each other by the following relationship:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

However, the above equation is seldom used to compute P. Instead P is directly solved from the Lyapunov equation.

Analysis of Linear Time Invariant Systems

- Choose an arbitrary symmetric positive definite matrix $Q \quad (Q = I)$
- Solve for the matrix P form the *Lyapunov equation* and verify whether it is positive definite
- Result: If P is positive definite, then $\dot{V}(X) < 0$ and hence the origin is "asymptotically stable".

Lyapunov's Indirect Theorem

Let the linearized system about $X = 0$ be $\Delta\dot{X} = A(\Delta X)$.

The theorem says that if all the eigenvalues $\lambda_i \ (i = 1, \ldots, n)$ of the matrix A satisfy $\text{Re}(\lambda_i) < 0$ (i.e. the linearized system is exponentially stable), then for the nonlinear system the origin is locally exponentially stable.

Instability theorem

Consider the autonomous dynamical system and assume $X=0$ is an equilibrium point. Let $V: D \rightarrow \mathbb{R}$ have the following properties:

$$(i)\, V(0)=0$$

$$(ii)\, \exists X_0 \in \mathbb{R}^n, \text{arbitrarily close to } X=0, \text{such that } V\left(X_0\right)>0$$

$$(iii)\; \dot{V}>0 \quad \forall X \in U, \text{ where the set } U \text{ is defined as follows}$$

$$U=\{X \in D: \|X\| \leq \varepsilon \;\text{ and }\; V\left(X\right)>0\}$$

Under these conditions, $X=0$ is unstable

Summary

- Motivation
- Notions of Stability
- Lyapunov Stability Theorems
- Stability Analysis of LTI Systems
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Thanks for the Attention….!