Lecture – 28

# Linear Quadratic Regulator (LQR) Design – II

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# Outline

- Stability and Robustness properties of LQR
- Optimum value of the cost function
- Extension of LQR design
  - For cross-product term in cost function
  - Rate of state minimization
  - Rate of control minimization
  - LQR design with prescribed degree of stability
- LQR for command tracking
- LQR for inhomogeneous systems

# Stability and Robustness Properties

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# LQR Design: Stability of Closed Loop System

- Closed loop system $\dot{X} = AX + BU = (A - BK)X$
- Lyapunov function $V(X) = X^T P X$

$$\dot{V} = \dot{X}^T P X + X^T P \dot{X}$$

$$= \left[(A - BK)X\right]^T PX + X^T P\left[(A - BK)X\right]$$

$$= X^T \left[\left(A - BR^{-1}B^T P\right)^T P + P\left(A - BR^{-1}B^T P\right)\right] X$$

$$= X^T \left[\left(PA + A^T P - PBR^{-1}B^T P + Q\right) - Q - PBR^{-1}B^T P\right] X$$

$$= X^T \left[-Q - PBR^{-1}B^T P\right] X$$

# LQR Design: Stability of Closed Loop System

For $R > 0, \quad R^{-1} > 0.$ Also $P > 0$

So $PBR^{-1}B^T P > 0$

Also $Q \geq 0.$

Hence, $\left( PBR^{-1}B^T P + Q \right) > 0$

$\therefore \ \dot{V}\left( X \right) < 0$

Hence, the closed loop system is always asymptotically stable!

# LQR Design: Minimum value of cost function

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left(X^T Q X + U^T R U\right) dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\left[X^T Q X + \left(-R^{-1}B^T P X\right)^T R\left(-R^{-1}B^T P X\right)\right] dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty} X^T\left(Q + PBR^{-1}B^T P\right) X \; dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\left(-\dot{V}\right) dt = -\frac{1}{2}\left[V\right]_{t_0}^{\infty} = -\frac{1}{2}\left[X^T P X\right]_{t_0}^{\infty}$$

$$= \frac{1}{2}\left[X_0^T P X_0 - X_\infty^T P X_\infty\right] = \frac{1}{2}\left(X_0^T P X_0\right)$$

# LQR Design: Robustness of Closed Loop System

- Gain Margin: $\infty$
- Phase Margin: $60^0$

(Ref.: D. S. Naidu, Optimal Control Systems, CRC Press, 2003.)

# Extensions of LQR Design

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# LQR Extensions:
# 1. Cross Product Term in P.I.

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left( X^T Q X + 2X^T W U + U^T R U \right) dt$$

Let us consider the expression:

$$X^T\left(Q - WR^{-1}W^T\right)X + \left(U + R^{-1}W^T X\right)^T R\left(U + R^{-1}W^T X\right)$$

$$= X^T Q X + U^T R U + \left(U^T W^T X + X^T W U\right)$$

$$= X^T Q X + 2X^T W U + U^T R U$$

# LQR Extensions:
# 1. Cross Product Term in P.I.

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left[ X^T \underbrace{\left(Q - WR^{-1}W^T\right)}_{Q_1} X + \left(U + R^{-1}W^T X\right)^T R \underbrace{\left(U + R^{-1}W^T X\right)}_{U_1} \right] dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\left( X^T Q_1 X + U_1 R U_1 \right) dt$$

$$\begin{aligned} \dot{X} &= AX + BU \\ &= AX + B\left(U_1 - R^{-1}W^T X\right) \\ &= \left(A - BR^{-1}W^T\right) X + BU_1 \\ &= A_1 X + BU_1 \end{aligned}$$

**Control Solution**

$$U_1 = -K\,X$$

$$\begin{aligned} U &= U_1 - R^{-1}W^T X \\ &= -\left(K + R^{-1}W^T\right) X \end{aligned}$$

# LQR Extensions:
# 2. Weightage on Rate of State

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left(X^T Q X + U^T R U + \dot{X}^T S \dot{X}\right)dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\left[X^T Q X + U^T R U + \left(AX + BU\right)^T S\left(AX + BU\right)\right]dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\begin{bmatrix} X^T Q X + U^T R U + X^T A^T S A X + X^T A^T S B U \\ + U^T B^T S A X + U^T B^T S B U \end{bmatrix}dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\left[X^T \overbrace{\left(Q + A^T S A\right)}^{Q_1} X + U^T \overbrace{\left(R + B^T S B\right)}^{R_1} U + 2X^T \overbrace{\left(A^T S B\right)}^{W} U\right]dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty}\left(X^T Q_1 X + U^T R_1 U + 2X^T W U\right)dt$$

⇒ Leads to a cross product case

# LQR Extensions:
# 3. Weightage on Rate of Control

$$J = \frac{1}{2}\int_0^\infty \left( X^T Q X + U^T R U + \dot{U}^T \hat{R} \dot{U} \right) dt$$

Let $\mathbf{X} = \begin{bmatrix} X \\ \hline U \end{bmatrix}, \quad V = \dot{U}$

$$J = \frac{1}{2}\int_0^\infty \left( \mathbf{X}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \mathbf{X} + V^T \hat{R} V \right) dt$$

$$J = \frac{1}{2}\int_0^\infty \left( \mathbf{X}^T \hat{Q} \, \mathbf{X} + V^T \hat{R} V \right) dt$$

# LQR Extensions:
# 3. Weightage on Rate of Control

$$\dot{X} = AX + BU, \quad X(0) = X_0$$

$$\dot{U} = V$$

$$\dot{\mathbf{X}} = \underbrace{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}}_{\hat{A}} \mathbf{X} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\hat{B}} V = \hat{A}\mathbf{X} + \hat{B}V$$

**Note:**

(1) The dimension of the problem has increased from $n$ to $(n+m)$

(2) If $\{A, B\}$ is controllable, it can be shown that the new system is also controllable.

# LQR Extensions:
# 3. Weightage on Rate of Control

**Solution :**

$$V = \dot{U} = -\hat{R}^{-1}\hat{B}^{T}\hat{P}\mathbf{X}$$

where $\hat{P}$ is the solution of

$$\hat{A}^{T}\hat{P} + \hat{P}A - \hat{P}\hat{B}\hat{R}^{-1}\hat{B}^{T}\hat{P} + \hat{Q} = 0$$

Hence

$$\dot{U} = -\hat{R}^{-1}\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^{\,T} & \hat{P}_{22} \end{bmatrix}\mathbf{X} = -\hat{R}^{-1}\begin{bmatrix} \hat{P}_{12}^{\,T} & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} X \\ U \end{bmatrix}$$

$$= -\hat{R}^{-1}\hat{P}_{12}^{\,T}X - \hat{R}^{-1}\hat{P}_{22}U$$

# LQR Extensions:
# 3. Weightage on Rate of Control

However, $\dot{U} = -\hat{R}^{-1}\hat{P}_{12}{}^T X - \hat{R}^{-1}\hat{P}_{22}U$ is a dynamic equation in $U$ and hence is not easy for implementation. For this reason, we want an expression in the RHS only as a function of $X$ and operations on it.

State equation: $\dot{X} = AX + BU$

This suggests: $U = B^{+}\left(\dot{X} - AX\right)$

(**Note :** This is only an approximate solution, unless $m \geq n$)

# LQR Extensions: 3. Weightage on Rate of Control

$$\dot{U} = -\hat{R}^{-1}\hat{P}_{12}{}^{T}X - \hat{R}^{-1}\hat{P}_{22}B^{+}\dot{X} - \hat{R}^{-1}\hat{P}_{22}B^{+}A\dot{X}$$

$$= -\underbrace{\hat{R}^{-1}\left(\hat{P}_{12}{}^{T} + \hat{P}_{22}B^{+}A\right)}_{K_2}X - \underbrace{\hat{R}^{-1}\hat{P}_{22}B^{+}}_{K_1}\dot{X}$$

$$= -K_1\dot{X} - K_2X$$

Integrating this expression both sides,

$$U = -\underbrace{K_1X}_{\text{Proportional}} - \underbrace{K_2\int_0^t X(z)\,dz}_{\text{Itegral}} + \underbrace{U_0}_{\text{Initial condition}}$$

**Note :** $U_0$ can be obtained using a performance index without the $\dot{U}$ term

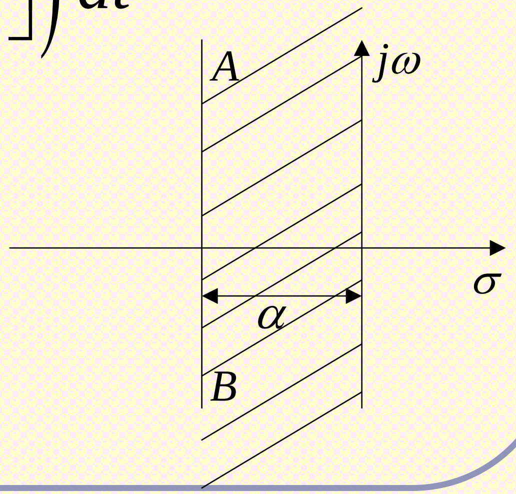
# LQR Extensions:
# 4. Prescribed Degree of Stability

**Condition:** All the Eiganvalues of the closed loop system should lie to the left of line *AB*

$$J = \frac{1}{2}\int_{t_0}^{\infty} e^{2\alpha t}\left[X^T Q X + U^T R U\right] dt \quad \text{where, } \alpha \geq 0$$

$$= \frac{1}{2}\int_{t_0}^{\infty} \left(\left[e^{\alpha t} X\right]^T Q\left[e^{\alpha t} X\right] + \left[e^{\alpha t} U\right]^T R\left[e^{\alpha t} U\right]\right) dt$$

$$= \frac{1}{2}\int_{t_0}^{\infty} \left(\tilde{X}^T Q \tilde{X} + \tilde{U}^T R \tilde{U}\right) dt$$




Let $\tilde{X} = e^{\alpha t} X$

$\tilde{U} = e^{\alpha t} U$

Co-ordinate transformation

# LQR Extensions:
# 4. Prescribed Degree of Stability

$$\dot{\tilde{X}} = e^{\alpha t}\dot{X} + \alpha e^{\alpha t}X$$

$$= e^{\alpha t}\left(AX + BU\right) + \alpha e^{\alpha t}X$$

$$= A\left(e^{\alpha t}X\right) + B\left(e^{\alpha t}U\right) + \alpha\left(e^{\alpha t}X\right)$$

$$\dot{\tilde{X}} = \left(A + \alpha I\right)\tilde{X} + B\tilde{U}$$

**Control Solution:**

$$\tilde{U} = -K\tilde{X}$$

$$e^{\alpha t}U = -Ke^{\alpha t}X$$

$$U = -KX$$

# LQR Extensions:
# 4. Prescribed Degree of Stability

**Modified System:**

$$\tilde{U} = -K\,\tilde{X}$$

$$\dot{\tilde{X}} = \left[\left(A - BK\right) + \alpha I\right]\tilde{X}$$

**Actual System:**

$$U = -K\,X$$

$$\dot{X} = \left(A - BK\right)X$$

$K$ is designed in such a way that eigenvalues of $\left[\left(A - BK\right) + \alpha I\right]$ will lie in the left-half plane.

Hence, eigenvalues of $\left(A - BK\right)$ will lie to the left of a line parallel to the imaginary axis, which is located away by distance $\alpha$ from the imaginary axis.

# LQR Design for Command Tracking

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# LQR Design for Command Tracking

**Problem:**

To design $U$ such that a part of the state vector of the linear system $\dot{X} = AX + BU$ tracks a commanded reference signal.

i.e. $X_T \rightarrow r_c$, where $X = \begin{bmatrix} X_T \\ \hline X_N \end{bmatrix}$

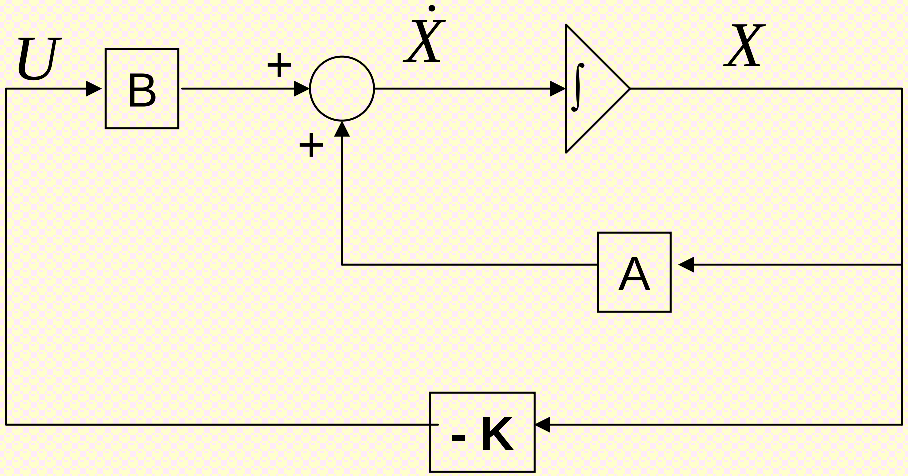
**Solution:**

1) Formulate a standard LQR problem.
However, select the $Q$ matrix properly.

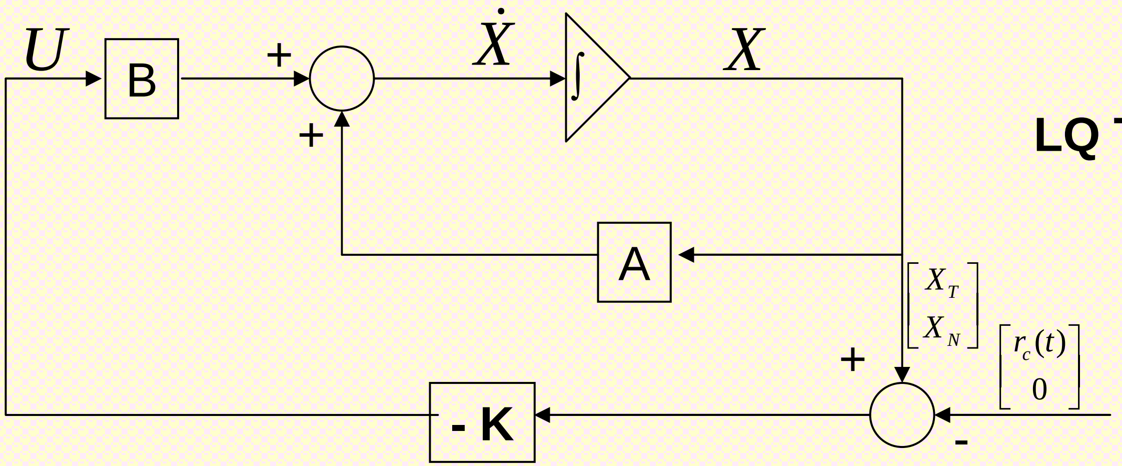
Typically $Q = \begin{bmatrix} Q_{TT} & 0 \\ 0 & 0 \end{bmatrix}$

2) Implement the controller as $U = -K \begin{bmatrix} X_T - r_c \\ X_N \end{bmatrix}$

# LQR Design for Command Tracking




**LQ Regulation**




**LQ Tracking**

# LQR Design for Command Tracking with Integral Feedback

**Solution (with integral controller):**

1) Augment the system dynamics with integral states

$$\begin{bmatrix} \dot{X}_T \\ \dot{X}_N \\ \dot{X}_I \end{bmatrix} = \begin{bmatrix} A_{TT} & A_{TN} & 0 \\ A_{NT} & A_{NN} & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} X_T \\ X_N \\ X_I \end{bmatrix} + \begin{bmatrix} B_T \\ B_N \\ 0 \end{bmatrix} U$$

2) Select the $Q$ matrix properly

(should penalize only $X_T$ and $X_I$ states)

3) Control solution $U = -K \left[ \left( X_T - r_c \right)^T \quad X_N^T \quad \left( \int_0^t \left( X_T - r_c \right) dt \right)^T \right]^T$

# LQR Design for Inhomogeneous Systems

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# LQR Design for Inhomogeneous Systems

**Reference**

**Optimum Intercept Laws for Accelerating Targets**

VITALIJ GARBER*

AIAA JOURNAL VOL. 6, NO. 11

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# LQR Design for Inhomogeneous Systems

- To derive the state $X$ of a linear (rather linearized) system $\boxed{\dot{X} = AX + BU + C}$ to the origin by minimizing the following quadratic performance index (cost function)

$$J = \frac{1}{2}\left(X_f^T S_f X_f\right) + \frac{1}{2}\int_{t_0}^{t_f}\left(X^T Q X + U^T R U\right) dt$$

where

$$S_f, Q \geq 0 \ \ \text{(psdf)}, \quad R > 0 \ \ \text{(pdf)}$$

# LQR Design for Inhomogeneous Systems

- Performance Index (to minimize):

$$J = \frac{1}{2}\left(X_f^T S_f X_f\right) + \frac{1}{2}\int_{t_0}^{t_f}\left(X^T Q X + U^T R U\right) dt$$

- Path Constraint: $\dot{X} = AX + BU + C$

- Boundary Conditions: $X(0) = X_0$ : Specified

  $t_f$ : Fixed, $X(t_f)$ : Free

# LQR Design for Inhomogeneous Systems

- Terminal penalty: $\varphi\left(X_f\right)=\frac{1}{2}\left(X_f^T S_f X_f\right)$
- Hamiltonian: $H=\frac{1}{2}\left(X^T Q X+U^T R U\right)+\lambda^T\left(AX+BU+C\right)$
- State Equation: $\dot{X}=AX+BU+C$
- Costate Equation: $\dot{\lambda}=-\left(\partial H/\partial X\right)=-\left(QX+A^T\lambda\right)$
- Optimal Control Eq.: $\left(\partial H/\partial U\right)=0 \quad\Rightarrow\quad U=-R^{-1}B^T\lambda$
- Boundary Condition: $\lambda_f=\left(\partial\varphi/\partial X_f\right)=S_f X_f$

# LQR Design for Inhomogeneous Systems

Guess $\boxed{\lambda(t) = P(t)\,X(t) + K(t)}$

$$\dot{\lambda} = \dot{P}X + P\dot{X} + \dot{K}$$

$$= \dot{P}X + P\left(AX + BU + C\right) + \dot{K}$$

$$= \dot{P}X + P\left(AX - BR^{-1}B^{T}\lambda\right) + PC + \dot{K}$$

$$-\left(QX + A^{T}\left(PX + K\right)\right) = \dot{P}X + P\left(AX - BR^{-1}B^{T}\left(PX + K\right)\right) + PC + \dot{K}$$

$$\left(\dot{P} + PA + A^{T}P - PBR^{-1}B^{T}P + Q\right)X$$

$$\left(\dot{K} + A^{T}K - PBR^{-1}B^{T}P + PC\right) = 0$$

# LQR Design for Inhomogeneous Systems

- Riccati equation

$$\dot{P} + PA + A^T P - PBR^{-1}B^T P + Q = 0$$

- Auxiliary equation

$$\dot{K} + \left( A^T - PBR^{-1}B^T \right) K + PC = 0$$

- Boundary conditions

$$P\left(t_f\right) X_f + K\left(t_f\right) = S_f X_f \quad \left( X_f \text{ is free} \right)$$

$$P\left(t_f\right) = S_f \qquad K\left(t_f\right) = 0$$

# LQR Design for Inhomogeneous Systems

**Control Solution:**

$$\begin{aligned} U &= -R^{-1}B^T\lambda \\ &= -R^{-1}B^T\left(PX + K\right) \\ &= -R^{-1}B^T PX - R^{-1}B^T K \end{aligned}$$

**Note:** There is a residual controller even after $X \to 0$.
This part of the controller offsets the continuous disturbance.

# Thanks for the Attention...!