Lecture – 26

Classical Numerical Methods to Solve Optimal Control Problems

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Necessary Conditions of Optimality in Optimal Control

- State Equation $\dot{X} = \frac{\partial H}{\partial \lambda} = f\left(t, X, U\right)$
- Costate Equation $\dot{\lambda} = -\left(\frac{\partial H}{\partial X}\right) = g\left(t, X, U\right)$
- Optimal Control Equation $\left(\frac{\partial H}{\partial U} = 0\right) \Rightarrow U = \psi\left(X, \lambda\right)$
- Boundary Condition $\lambda_f = \frac{\partial \varphi}{\partial X_f}$ $X\left(t_0\right) = X_0 : \text{Fixed}$

Necessary Conditions of Optimality: Salient Features

- State and Costate equations are dynamic equations
- State equation develops forward whereas Costate equation develops backwards
- Optimal control equation is a stationary equation
- The formulation leads to Two-Point-Boundary-Value Problems (TPBVPs), which demand computationally-intensive iterative numerical procedures to obtain the optimal control solution

Classical Methods to Solve TPBVPs

- Gradient Method
- Shooting Method
- Quasi-Linearization Method

Gradient Method

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Gradient Method

- Assumptions:
  - State equation satisfied
  - Costate equation satisfied
  - Boundary conditions satisfied
- Strategy:
  - Satisfy the optimal control equation

Gradient Method

$$\delta \bar{J} = \left(\delta X_f\right)^T \left[\frac{\partial \phi}{\partial X_f} - \lambda_f\right]$$

$$+\int_{t_0}^{t_f} \left(\delta X\right)^T \left[\frac{\partial H}{\partial X} + \dot{\lambda}\right] dt$$

$$+\int_{t_0}^{t_f} \left(\delta U\right)^T \left[\frac{\partial H}{\partial U}\right] dt$$

$$+\int_{t_0}^{t_f} \left(\delta \lambda\right)^T \left[\frac{\partial H}{\partial \lambda} - \dot{X}\right] dt$$

Gradient Method

- After satisfying the state & costate equations and boundary conditions, we have

$$\delta \bar{J} = \int_{t_0}^{t_f} (\delta U)^T \left[\frac{\partial H}{\partial U} \right] dt$$

- Select

$$\delta U(t) = -\tau \left[\frac{\partial H}{\partial U} \right], \quad \tau > 0$$

- This leads to

$$\delta \bar{J} = -\tau \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial U} \right]^T \left[\frac{\partial H}{\partial U} \right] dt$$

Gradient Method

- We select $$\delta U^i(t) = \left[U^{i+1}(t) - U^i(t) \right] = -\tau \left[\frac{\partial H}{\partial U} \right]^i$$

- This lead to $$U^{i+1}(t) = U^i(t) - \tau \left[\frac{\partial H}{\partial U} \right]^i$$

- Note: $$\delta \bar{J} = -\tau \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial U} \right]^T \left[\frac{\partial H}{\partial U} \right] dt \le 0$$

- Eventually, $$\delta \bar{J} = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial U} = 0$$

Gradient Method: Procedure

- Assume a control history (not a trivial task)
- Integrate the state equation forward
- Integrate the costate equation backward
- Update the control solution
  - This can either be done at each step while integrating the costate equation backward or after the integration of the costate equation is complete
- Repeat the procedure until convergence

$$\int_{t_0}^{t_f} \left[\frac{\partial H}{\partial U}\right]^T \left[\frac{\partial H}{\partial U}\right] dt \le \gamma \qquad \text{(a pre-selected constant)}$$

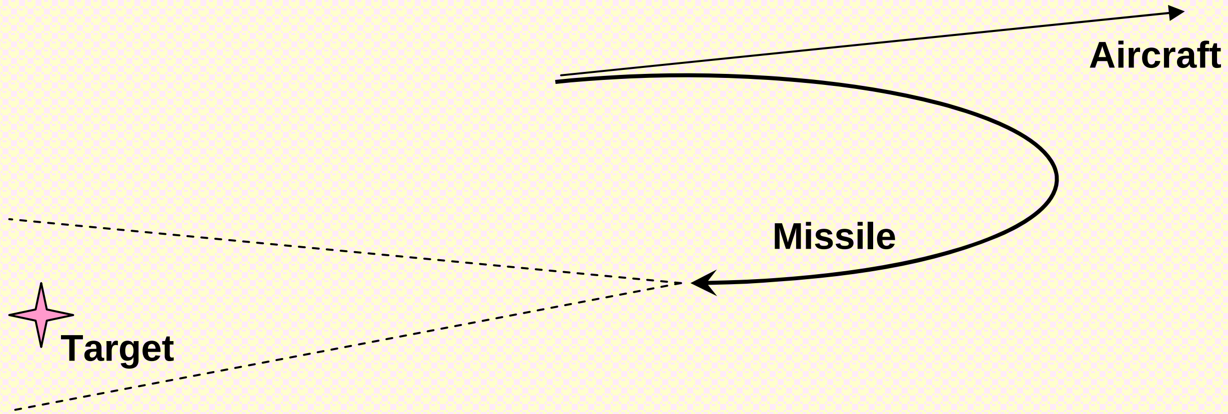
Gradient Method: Selection of τ

- Select τ so that it leads to a certain percentage reduction of $\bar{J}$
- Let the percentage be α
- Then $$\tau \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial U}\right]^T \left[\frac{\partial H}{\partial U}\right] dt = \frac{\alpha}{100}\left|\bar{J}\right|$$
- This leads to $$\boxed{\tau = \frac{\frac{\alpha}{100}\left|\bar{J}\right|}{\int_{t_0}^{t_f} \left[\frac{\partial H}{\partial U}\right]^T \left[\frac{\partial H}{\partial U}\right] dt}}$$

A Real-Life Challenging Problem

Objective:

Air-to-air missiles are usually launched from an aircraft in the forward direction. However, the missile should turn around and intercept a target "behind the aircraft".



To execute this task, the missile should turn around by -180^o and lock onto its target (after that it can be guided by its own homing guidance logic).

Note: Every other case can be considered as a subset of this extreme scenario!

A Real-Life Challenging Problem

MATHEMATICAL PERSPECTIVE:

- Minimum time optimization problem
- Fixed initial conditions and free final time problem

SYSTEM DYNAMICS:

Equations of motion for a missile in vertical plane. The non-dimensional equations of motion (point mass) in a vertical plane are:

$$M' = -S_w M^2 C_D - \sin(\gamma) + T_w \cos(\alpha)$$

$$\gamma' = \frac{1}{M}[S_w M^2 C_L + T_w \sin(\alpha) - \cos(\gamma)]$$

where prime denotes differentiation with respect to the non-dimensional time τ

A Real-Life Challenging Problem

The non-dimensional parameters are defined as follows:

$$\tau = \frac{g}{at}; \quad T_w = \frac{T}{mg}; \quad S_w = \frac{\rho a^2 S}{2mg}; \quad M = \frac{V}{a}$$

where M = flight Mach number

- γ = flight path angle
- m = mass of the missile
- V = speed of the missile
- C_D = drag coefficient
- a = the local speed of sound
- t = flight time after launch
- T = thrust
- S = reference aerodynamic area
- C_L = lift coefficient
- g = the acceleration due to gravity
- ρ = the atmospheric density

NOTE: C_L, C_D are usually functions of α & M (tabulated data)

A Real-Life Challenging Problem

COST FUNCTION:

Mathematically the problem is possed as follows to find the control minimizing cost function:

$$J = \int_0^{t_f} dt$$

Constraints $\gamma(0) = 0^o$, $M(0) =$ initial Mach number

$$\gamma(t_f) = -180^o, \; M(t_f) = 0.8$$

A Real-Life Challenging Problem

Choosing γ as the independent variable the equations are reformulated as follows:

$$\frac{dM}{d\gamma} = \frac{\left(-S_w M^2 C_D - \sin(\gamma) + T_w \cos(\alpha)\right) M}{S_w M^2 C_L - \cos(\gamma) + T_w \sin(\alpha)}$$

$$\frac{dt}{d\gamma} = \frac{a\,M}{g\left(S_w M^2 C_L - \cos(\gamma) + T_w \sin(\alpha)\right)}$$

and the transformed cost function is

$$J = \int_0^{t_f} dt = \int_0^{-\pi} \frac{dt}{d\gamma}\, d\gamma = \int_0^{-\pi} \frac{a\,M}{g\left(S_w M^2 C_L - \cos(\gamma) + T_w \sin(\alpha)\right)}\, d\gamma$$

(A difficult minimum-time problem has been converted to a relatively easier fixed final-time problem (with hard constraint: $M(\gamma_f) = 0.8$)!)

Task

Solve the problem using gradient method. Assume $M(0) = 0.5$ and engagement height as 5 km. Next, generate the trajectories and tabulate the values of M_f for various q values.

Use the following system parameters

(typical for an air-to-air missile):

$m = 240\ kg$

$S = 0.0707\ m2$

$T = 24{,}000\ N$

$C_D = 0.5$

$C_L = 3.12$

Use standard atmosphere chart for the atmospheric data.

Shooting Method

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Necessary Conditions of Optimality (TPBVP): A Summary

- State Equation $\dot{X} = \dfrac{\partial H}{\partial \lambda} = f(t, X, U)$
- Costate Equation $\dot{\lambda} = -\left(\dfrac{\partial H}{\partial X}\right) = g(t, X, U, \lambda)$
- Optimal Control Equation $\dfrac{\partial H}{\partial U} = 0$
- Boundary Condition $\lambda_f = \dfrac{\partial \varphi}{\partial X_f}$ $X(t_0) = X_0 : \text{Fixed}$

Shooting Method

- Form a *Meta State Vector* $Z = \begin{bmatrix} X \\ \lambda \end{bmatrix}$. This implies $dZ = \begin{bmatrix} dX \\ d\lambda \end{bmatrix}$.

- Guess $\lambda(t_0)$. Note that $X(t_0)$ is given. This leads to

$$\begin{aligned} \dot{Z} &\equiv \begin{bmatrix} \dot{X} \\ \dot{\lambda} \end{bmatrix} = F(Z) \qquad (1) \\ Z(t_0) &= \begin{bmatrix} X(t_0) \\ \lambda(t_0) \end{bmatrix} \end{aligned}$$

- Obtain the linearized *Error Dynamics Equation*

$$d\dot{Z} = \left[\frac{\partial F}{\partial Z}\right] dZ \qquad (2)$$

Shooting Method

- Define a *State Transition Matrix (STM)* Φ, such that at any two times t_i and t_j,

$$dZ(t_j) = \Phi(t_j, t_i) \quad dZ(t_i) \tag{3}$$

- The dynamics and initial conditions for the STM can be shown to be

$$\begin{aligned} \dot{\Phi} &= \left[\frac{\partial F}{\partial Z}\right] \Phi \\ \Phi(t_0, t_0) &= I_{2n \times 2n} \end{aligned} \tag{4}$$

- Numerically integrate the equations (2) and (4) from t_0 to t_f; solving for the *optimal control* U at each instant of time.

Shooting Method

- Finally, at $t = t_f$,

$$dZ_f \equiv \begin{bmatrix} dX_f \\ d\lambda_f \end{bmatrix} = \Phi(t_f, t_0) \quad dZ_0 \tag{5}$$

- Thus, at $t = t_0$,

$$dZ_0 \equiv \begin{bmatrix} dX_0 \\ d\lambda_0 \end{bmatrix} = \Phi^{-1}(t_f, t_0) \quad dZ_f \tag{6}$$

- Since X_0 is fixed, force $dX_0 = 0$. Update only λ_0. Repeat until *convergence*.

Shooting Method

- **Computational Load Reduction**

Partition the STM (Φ) as $\Phi = \left[\Phi_1 \;\; \vdots \;\; \Phi_2 \right]$. Then,

$$dZ_f = \Phi_{1f}\; dX_0 + \Phi_{2f}\; d\lambda_0 = \Phi_{2f}\; d\lambda_0 \tag{7}$$

Shooting Method

- For convenience, let $h = (\lambda_f)_{n\times 1}$ be the vector of n boundary conditions at t_f. Then,

$$\left(\frac{\partial h}{\partial Z}\right)_f dZ_f = dh = \left(\lambda_f \ - \lambda_f^*\right)_{n\times 1} \tag{8}$$

Where, λ_f^* is the true (desired) value of λ_f.

- Finally, at $t = t_f$,

$$d\lambda_0 \quad = \quad \left[\left(\frac{\partial h}{\partial Z}\right)_f \Phi_{2f}\right]^{-1} dh \tag{9}$$

- Hence, obtain $d\lambda_0(k)$ and update $\lambda_0(k)$ to $\lambda_0(k+1)$. Repeat until convergence.

Quasi-Linearization Method

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Quasi-Linearization Method

Problem:

Differential Equation: $\dot{Z} = F(Z, t), \qquad Z \triangleq \begin{bmatrix} X^T & \lambda^T \end{bmatrix}^T$

Boundary condition: $\langle C(t_i), Z(t_i) \rangle = C_i^T Z_i = b_i$

$$t_i \in t, \quad i \in \{1, \ldots, n\}$$

Assumption:

This vector differential equation has a unique solution over $t \in [t_0, t_f]$

Trick:

The nonlinear multi-point boundary value problem is transformed into a sequence of linear non-stationary boundary value problems, the solution of which is made to approximate the solution of the true problem.

Quasi-Linearization Method

(1) Guess an approximate solution $Z^N(t)$ $(N=1)$ (it need not satisfy the B.C.)

For updating this solution, proceed with the following steps:

(2) Linearize the system dynamics about $Z^N(t)$

$$\Delta \dot{Z}^N = \underbrace{\left[\frac{\partial F}{\partial Z}\right]_{Z^N}}_{A(t)} \Delta Z^N, \quad \text{where, } \Delta Z^N(t) \triangleq \underbrace{Z^{N+1}(t)}_{\text{To be found}} - Z^N(t)$$

$$\Delta \dot{Z}^N = A(t)\,\Delta Z^N$$

(3) Enforce the boundary with respect to the updated solution $Z^{N+1}(t)$

$$\left\langle C(t_i), Z^{N+1}(t_i)\right\rangle = \left\langle C(t_i),\; Z^N(t_i) + \Delta Z^N(t_i)\right\rangle = b_i$$

$$\left\langle C(t_i), \Delta Z^N(t_i)\right\rangle = -\left\langle C(t_i), Z^N(t_i)\right\rangle + b_i$$

Philosophy: Solve this linear system and update the solution!

Quasi-Linearization Method: Solution by STM Approach

(1) From the linearized system dynamics, we can write

$$\dot{Z}^{N+1} = \dot{Z}^{N} + A(t)\left(Z^{N+1} - Z^{N}\right)$$

$$= \underbrace{A(t)\,Z^{N+1}}_{\text{Homogeneous}} + \underbrace{\left[F\left(Z^{N},t\right) - A(t)\,Z^{N}\right]}_{\text{Forcing function}}$$

(2) The solution $Z^{N+1}(t)$ to the above equation is given by

$$Z^{N+1}(t) = \underbrace{\Phi^{N+1}(t, t_0)}_{\text{State transition matrix (STM)}} \; Z^{N+1}(t_0) + \underbrace{p^{N+1}(t)}_{\text{Particular solution}}$$

(3) The solution for STM $\Phi^{N+1}(t, t_0)$ can be obtained from the fact that it satisfies the following differential equation and boundary conditions

$$\frac{\partial}{\partial t}\left[\Phi^{N+1}(t, t_0)\right] = A(t)\;\Phi^{N+1}(t, t_0)$$

$$\Phi^{N+1}(t_0, t_0) = I$$

Quasi-Linearization Method: Solution by STM Approach

(4) The particular solution $p^{N+1}(t)$ can be obtained by observing that it satisfies the the following differential equation and boundary condition

Substituting the complete solution $Z^{N+1}(t)$ in the original equation

$$\frac{\partial}{\partial t}\left[\Phi^{N+1}(t,t_0)Z^{N+1}(t_0)\right]+\dot{p}^{N+1}(t)= A(t)\left[\Phi^{N+1}(t,t_0)Z^{N+1}(t_0)+p^{N+1}(t)\right] + \left[F\left(Z^N,t\right)-A(t)Z^N\right]$$

$$\dot{p}^{N+1}(t)= A(t)\,p^{N+1}(t)+\left[F\left(Z^N,t\right)-A(t)Z^N\right]$$

(5) The boundary condition $p^{N+1}(t_0)$ can be obtained by observing that

$$Z^{N+1}(t_0)=\underbrace{\Phi^{N+1}(t_0,t_0)}_{I}Z^{N+1}(t_0)+p^{N+1}(t_0)$$

$$p^{N+1}(t_0)=0$$

Quasi-Linearization Method: Solution by STM Approach

(6) The boundary condition $Z^{N+1}(t_0)$ can be obtained as follows

$$\left\langle C(t_i),\ Z^{N+1}(t_i)\right\rangle = b_i$$

$$\left\langle C(t_i),\ \Phi^{N+1}(t_i,t_0)Z^{N+1}(t_0)+p^{N+1}(t_i)\right\rangle = b_i$$

$$\left\langle C(t_i),\ \Phi^{N+1}(t_i,t_0)Z^{N+1}(t_0)\right\rangle = -\left\langle C(t_i),\ p^{N+1}(t_i)\right\rangle + b_i$$

Solve the above system to obtain $Z^{N+1}(t_0)$

Once $Z^{N+1}(t_0)$ is determined, the solution $Z^{N+1}(t)$ is available from the

STM solution: $$Z^{N+1}(t) = \underbrace{\Phi^{N+1}(t,t_0)}_{\text{STM}} Z^{N+1}(t_0) + \underbrace{p^{N+1}(t)}_{\text{Particular solution}}$$

Quasi-Linearization Method: Convergence Property

Under the assumption that the problem admits a unique solution for $t \in \left[t_0, t_f\right]$ it can be shown that the sequence of vectors $\left\{Z^{N+1}(t)\right\}$ converge to the true solution.

Morover, the process can be shown to have "quadratic convergence" in general i.e., it can be shown that $\left\| Z^{N+1}(t) - Z^{N}(t) \right\| \leq k \left\| Z^{N}(t) - Z^{N-1}(t) \right\|$, where $k \neq f(N)$.

Further more, for a large class of systems, it can be shown to have "monotone convergence" as well, i.e. there won't be any over-shooting in the convergence process.

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A Demonstrative Example

Problem: Minimize $J = \frac{1}{2}\int_0^1 \left(x^2 + u^2\right) dt$ for the system $\dot{x} = -x^2 + u, \quad x(0) = 10.$

Solution:

Hamiltonian: $H = \frac{1}{2}\left(x^2 + u^2\right) + \lambda\left(-x^2 + u\right)$

1) State Equation: $\dot{x} = -x^2 + u$

2) Optimal Control Equation: $u + \lambda = 0 \quad \Rightarrow \quad u = -\lambda$

3) Costate Equation: $\dot{\lambda} = -\left(\partial H / \partial x\right) = -x + 2\lambda x$

4) Bounary Conditions: $x(0) = 10, \quad \lambda(1) = \left(\partial \Phi / \partial x\right) = 0$

Substituting the expression for u in the state equation, we can write

$$\dot{x} = -x^2 - \lambda, \qquad x(0) = 10$$

$$\dot{\lambda} = -x + 2\lambda x, \qquad \lambda(1) = 0$$

Task : Solve this problem using shooting and quasi-linearization methods.

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Thanks for the Attention...!