Lecture – 25

Optimal Control Formulation using Calculus of Variations

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Topics

Optimal Control Formulation

- Objective & Selection of Performance Index
- Necessary Conditions of Optimality and Two-Point Boundary Value Problem (TPBVP) Formulation
- Boundary/Transversality Conditions
- Numerical Examples

Optimal Control Formulation: Objective & Selection of Performance Index

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Objective

To find an "admissible" time history of control variable $U(t), t \in [t_0, t_f]$ which:

1) Causes the system giverned by $\dot{X} = f(t, X, U)$
to follow an admissible trajectory

2) Optimizes (minimizes/maximizes) a "meanigful" performance index

$$J = \varphi(t_f, X_f) + \int_{t_0}^{t_f} L(t, X, U)\, dt$$

3) Forces the system to satisfy "proper boundary conditions".
[our focus: $X(t_0) = X_0$ (given), t_f : fixed]

Meaningful Performance Index

1) Minimize the operational time

$$J = \left(t_f - t_0\right) = \int_{t_0}^{t_f} 1\, dt \qquad \left[\varphi = 0, \quad L = 1\right]$$

2) Minimize the control effort

$$J = \frac{1}{2}\int_{t_0}^{t_f} U^T R U \; dt\,, \quad R > 0 \qquad \left[\varphi = 0, \quad L = \frac{1}{2} U^T R U\right]$$

3) Minimize deviation of state from a fixed value C with minimum control effort

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[\left(X - C\right)^T Q \left(X - C\right) + U^T R U\right] dt, \quad Q \geq 0, R > 0$$

Meaningful Performance Index

4) Minimize deviation of state from origin with minimum control effort

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[X^T Q X + U^T R U \right] dt, \quad Q \ge 0, R > 0$$

5) Minimize the control effort, while the final state X_f reaches close to a constant C

$$J = \frac{1}{2}\left(X_f - C\right)^T S_f \left(X_f - C\right) + \frac{1}{2}\int_{t_0}^{t_f} \left(U^T R U\right) dt, \quad S_f \ge 0, R > 0$$

Optimal Control Using Calculus of Variations: Hamiltonian Formulation

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Optimal Control Problem

- Performance Index (to minimize / maximize):

$$J = \varphi\left(t_f, X_f\right) + \int_{t_0}^{t_f} L\left(t, X, U\right) dt$$

- Path Constraint:

$$\dot{X} = f\left(t, X, U\right)$$

- Boundary Conditions: $X\left(0\right) = X_0$: Specified

t_f : Fixed, $X\left(t_f\right)$: Free

Necessary Conditions of Optimality

- Augmented PI
$$\bar{J} = \varphi + \int_{t_0}^{t_f} \left[L + \lambda^T \left(f - \dot{X} \right) \right] dt$$
- Hamiltonian
$$H \triangleq \left(L + \lambda^T f \right)$$
- First Variation
$$\delta \bar{J} = \delta\varphi + \delta \int_{t_0}^{t_f} \left(H - \lambda^T \dot{X} \right) dt$$
$$= \delta\varphi + \int_{t_0}^{t_f} \delta \left(H - \lambda^T \dot{X} \right) dt$$

Necessary Conditions of Optimality

- First Variation $$\delta \bar{J} = \delta\varphi + \int_{t_0}^{t_f} \left(\delta H - \delta\lambda^T \dot{X} - \lambda^T \delta \dot{X} \right) dt$$

- Individual terms

$$\delta\varphi\left(t_f, X_f\right) = \left(\delta X_f\right)^T \left(\frac{\partial \varphi}{\partial X_f}\right)$$

$$\delta H\left(t, X, U, \lambda\right) = \left(\delta X\right)^T \left(\frac{\partial H}{\partial X}\right) + \left(\delta U\right)^T \left(\frac{\partial H}{\partial U}\right) + \left(\delta\lambda\right)^T \left(\frac{\partial H}{\partial \lambda}\right)$$

Necessary Conditions of Optimality

$$\int_{t_0}^{t_f} \left(\lambda^T \, \delta \dot{X} \right) dt = \int_{t_0}^{t_f} \left(\lambda^T \, \frac{d\left(\delta X\right)}{dt} \right) dt$$

$$= \left[\lambda^T \, \delta X \right]_{t_0, \delta X_0}^{t_f, \delta X_f} - \int_{t_0}^{t_f} \left(\frac{d\lambda}{dt} \right)^T \delta X \, dt$$

$$= \left[\lambda_f^T \, \delta X_f - \lambda_0^T \, \underbrace{\delta X_0}_{\mathbf{0}} \right] - \int_{t_0}^{t_f} \left(\delta X \right)^T \dot{\lambda}^T \, dt$$

$$= \lambda_f^T \, \delta X_f - \int_{t_0}^{t_f} \left(\delta X \right)^T \dot{\lambda}^T \, dt$$

Necessary Conditions of Optimality

- First Variation

$$\delta \bar{J} = \left(\delta X_f\right)^T \left(\frac{\partial \varphi}{\partial X_f}\right) - \left(\delta X_f\right)^T \lambda_f$$

$$+ \int_{t_0}^{t_f} \left[\left(\delta X\right)^T \left(\frac{\partial H}{\partial X}\right) + \left(\delta U\right)^T \left(\frac{\partial H}{\partial U}\right) + \left(\delta \lambda\right)^T \left(\frac{\partial H}{\partial \lambda}\right) \right] dt$$

$$+ \int_{t_0}^{t_f} \left(\delta X\right)^T \dot{\lambda}\, dt - \int_{t_0}^{t_f} \left(\delta \lambda\right)^T \dot{X}\, dt$$

Necessary Conditions of Optimality

- First Variation

$$\delta \bar{J} = \left(\delta X_f\right)^T \left[\frac{\partial \varphi}{\partial X_f} - \lambda_f\right]$$

$$+ \int_{t_0}^{t_f} \left(\delta X\right)^T \left[\frac{\partial H}{\partial X} + \dot{\lambda}\right] dt + \int_{t_0}^{t_f} \left(\delta U\right)^T \left[\frac{\partial H}{\partial U}\right] dt$$

$$+ \int_{t_0}^{t_f} \left(\delta \lambda\right)^T \left[\frac{\partial H}{\partial \lambda} - \dot{X}\right] dt$$

= 0

Necessary Conditions of Optimality: Summary

- State Equation $$\dot{X} = \frac{\partial H}{\partial \lambda} = f\left(t, X, U\right)$$
- Costate Equation $$\dot{\lambda} = -\left(\frac{\partial H}{\partial X}\right)$$
- Optimal Control Equation $$\frac{\partial H}{\partial U} = 0$$
- Boundary Condition $$\lambda_f = \frac{\partial \varphi}{\partial X_f} \qquad X\left(t_0\right) = X_0 : \text{Fixed}$$

Necessary Conditions of Optimality: Some Comments

- State and Costate equations are dynamic equations
- Optimal control equation is a stationary equation
- Boundary conditions are split: it leads to **Two-Point-Boundary-Value Problem (TPBVP)**
- State equation develops forward whereas Costate equation develops backwards
- Traditionally, TPBVPs demand computationally-intensive iterative numerical procedures
- These iterative numerical procedures lead to “open-loop” control solutions

An Useful Theorem

Theorem:

If the Hamiltonian H is not an explicit function of time, then H is 'constant' along the optimal path.

Proof:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \dot{X}^T \frac{\partial H}{\partial X} + \dot{U}^T \frac{\partial H}{\partial U} + \dot{\lambda}^T \frac{\partial H}{\partial \lambda}$$

$$= \frac{\partial H}{\partial t} + \dot{X}^T \underbrace{\left(\frac{\partial H}{\partial X} + \dot{\lambda} \right)}_{0} + \dot{U}^T \underbrace{\left(\frac{\partial H}{\partial U} \right)}_{0} \qquad \left(\because \quad \frac{\partial H}{\partial \lambda} = \dot{X} \quad \text{and} \quad \dot{\lambda}^T \dot{X} = \dot{X}^T \dot{\lambda} \right)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \quad (\text{on optimal path})$$

$$= 0 \qquad (\text{if } H \text{ is not an explicit function of } t). \text{ Hence, the result!}$$

General Boundary/Transversality Condition

General condition:

$$\left[\text{with } \left(t_0, X_0\right) \text{fixed}\right]$$

$$\left[\frac{\partial\Phi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f + \left[\frac{\partial\Phi}{\partial t} + H\right]_{t_f} \delta t_f = 0$$

Special Cases: 1) t_f : fixed, X_f : free

$$\left[\frac{\partial\Phi}{\partial X} - \lambda\right]_{t_f}^T \delta X_f = 0 \qquad \Rightarrow \qquad \lambda_f = \frac{\partial\Phi\left(t_f, X_f\right)}{\partial X_f}$$

2) t_f : free, X_f : fixed

$$\left[\frac{\partial\Phi}{\partial t} + H\right]_{t_f} \delta t_f = 0 \qquad \Rightarrow \qquad H\left(t_f\right) = \frac{\partial\Phi}{\partial t_f}$$

Example – 1: A Toy Problem

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Example

Problem:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 + u \end{bmatrix}$$

$$J = \frac{1}{2}\left(x_{1_f} - 5\right)^2 + \frac{1}{2}\left(x_{2_f} - 2\right)^2 + \frac{1}{2}\int_{t_0}^{t_f} u^2 \, dt$$

$$t_0 = 0, t_f = 2, \quad x_1(0) = x_2(0) = 0$$

Solution:

$$H = \left(u^2 / 2\right) + \lambda_1 x_2 + \lambda_2 \left(-x_2 + u\right)$$

Costate Eq. $\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -\left(\partial H / \partial x_1\right) \\ -\left(\partial H / \partial x_2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 + \lambda_2 \end{bmatrix}$

Optimal control Eq. $u + \lambda_2 = 0 \quad \Rightarrow \quad \boxed{u = -\lambda_2}$

Example

Boundary Conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1(2) \\ \lambda_2(2) \end{bmatrix} = \begin{bmatrix} x_1(2)-5 \\ x_2(2)-2 \end{bmatrix}$$

Define $\quad Z \triangleq \begin{bmatrix} x_1 & x_2 & \lambda_1 & \lambda_2 \end{bmatrix}^T$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\dot{Z} = AZ$$

Solution

$$Z(t) = e^{At} C$$

Example

Use the boundary condition at $t = 0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Use the boundary condition at $t_f = 2$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ x_1(2)-5 \\ x_2(2)-2 \end{bmatrix} = e^{2A} \begin{bmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0.86 & 1.63 & -2.76 \\ 0 & 0.14 & 2.76 & -3.63 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6.39 & 7.39 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{bmatrix}$$

Example

Four equations and four unknowns:

$$\begin{bmatrix} 1 & 0 & -1.63 & 2.76 \\ 0 & 1 & -2.76 & 3.63 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 6.39 & -7.39 \end{bmatrix} \begin{bmatrix} x_1(2) \\ x_2(2) \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1.63 & 2.76 \\ 0 & 1 & -2.76 & 3.63 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 6.39 & -7.39 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.30 \\ 1.33 \\ -2.70 \\ -2.42 \end{bmatrix}$$

Example

- Solution for State and Costate

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 0 \\ -2.70 \\ -2.42 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Solution for Optimal Control

$$u = -\lambda_2(t)$$

Example – 2: Double Integrator Problem

(Relevance: Satellite Attitude Control Problem)

$$\ddot{x} = u$$

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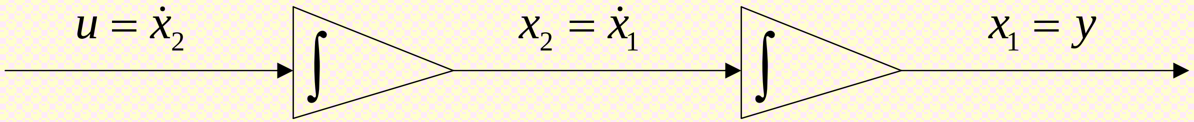
Double Integrator Problem



Consider a double integrator problem as shown in the above figure.

Find such $u(t)$ that the system initial values $X(0) = \begin{bmatrix} 10 & 0 \end{bmatrix}^T$ are driven to the origin by minimizing

$$J = t_f^{\ 2} + \frac{1}{2}\int_0^{t_f} u^2 dt$$

Note : (1) t_f : unspecified

(2) Control variable $u(t)$ is unconstrained

Double Integrator Problem

Solution :

System dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} U = AX + BU$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = CX \qquad \text{(not required)}$$

Boundary Condition

$$X(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad X(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Double Integrator Problem

Controllability Check :

Controllability Matrix

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\left| M \right| = -1 \neq 0$$

Hence, the system is controllable.

Necessary Conditions of Optimality

$$H = \frac{1}{2}u^2 + \lambda^T \left(AX + Bu \right)$$

(1) State Eq: $\dot{X} = AX + Bu$

(2) Optimal Control Eq: $\dfrac{\partial H}{\partial u} = 0$

$$u + B^T \lambda = 0$$

$$u = -B^T \lambda = -\lambda_2$$

(3) Costate Eq: $\dot{\lambda} = -\dfrac{\partial H}{\partial X} = -A^T \lambda$

Optimal Control Solution

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -A^T \lambda = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

$$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = c_1$$

$$\dot{\lambda}_2 = -\lambda_1 = -c_1$$

$$\lambda_2 = -c_1 t + c_2$$

$$\therefore \quad u = -\lambda_2 = c_1 t - c_2$$

Optimal State Solution

However,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} x_2 \\ c_1 t - c_2 \end{bmatrix}$$

Hence

$$x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3$$

$$x_1 = \int x_2 dt = c_1 \frac{t^3}{6} - c_2 \frac{t^2}{2} + c_3 t + c_4$$

Optimal State Solution

Using the B.C. at $t = 0$:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{6}t^3 - \frac{c_2}{2}t^2 + 10 \\ \frac{c_1}{2}t^2 - c_2 t \end{bmatrix}$$

Using the B.C at $t = t_f$:

$$\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \begin{bmatrix} \frac{c_1}{6}{t_f}^3 - \frac{c_2}{2}{t_f}^2 + 10 \\ \frac{c_1}{2}{t_f}^2 - c_2 t_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Transversality Conditions (t_f : free)

$$\left.\frac{\partial \varphi}{\partial t}\right|_{t_f} = -H\big|_{t_f}$$

$$2t_f = -\left[\frac{u^2}{2} + \lambda^T \left(AX + Bu\right)\right]_{t_f}$$

$$= -\left[\frac{u^2}{2} + \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}\begin{bmatrix}x_2 \\ u\end{bmatrix}\right]_{t_f}$$

$$= -\left[\frac{\left(c_1 t_f - c_2\right)^2}{2} + \lambda_1\left(t_f\right) x_2\left(t_f\right) - \left(c_1 t_f - c_2\right)^2\right]$$

$x_2(t_f) \rightarrow$ 0 (B.C.)

$$= \frac{1}{2}\left(c_1 t_f - c_2\right)^2$$

$$4t_f = c_1^{\,2} t_f^{\,2} - 2c_1 c_2 t_f + c_2^{\,2}$$

Transversality Conditions (t_f : free)

In summary, we have to solve for c_1, c_2 and t_f from:

$$c_1 t_f^{\ 3} - 3c_2 t_f^{\ 2} + 60 = 0$$

$$c_1 t_f^{\ 2} - 2c_2 t_f = 0$$

$$c_1^2 t_f^{\ 2} - \left(2c_1c_2 + 4t_f\right) + c_2^{\ 2} = 0$$

At this point, one can solve c_1, c_2 from first two equations in terms of t_f and subtitute them in the third equation. Then the resulting nonlinear equation in t_f can be solved (preferably in closed form). However, one must discard unrealistic solutions (e.g. $t_f \le 0$ is unrealistic).

Note : One may use numerical tehniques (like Newton-Raphson technique)

Transversality Conditions (t_f : free)

Finally, $\begin{bmatrix} c_1 \\ c_2 \\ t_f \end{bmatrix} = \begin{bmatrix} 2.025 \\ 3.95 \\ c_2^{\,2}/4 \end{bmatrix}$

Hence, the optimal solution is given by:

$$u = c_1 t - c_2 = 2.025t - 3.95$$

and it will take $t_f = \dfrac{(3.95)^2}{4} = 3.901$ time units to reach $X_f = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$,

starting from $X(0) = \begin{bmatrix} 10 & 0 \end{bmatrix}^T$

Note: (1) It is an open-loop control law

(2) The application of control has to be terminated at t_f

References on Optimal Control Design

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Thanks for the Attention...!